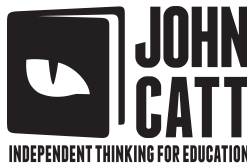


REFLECT, EXPECT, CHECK, EXPLAIN

SEQUENCES AND BEHAVIOUR TO ENABLE
MATHEMATICAL THINKING IN THE CLASSROOM

BY CRAIG BARTON



First Published 2020

by John Catt Educational Ltd,
15 Riduna Park, Station Road,
Melton, Woodbridge IP12 1QT

Tel: +44 (0) 1394 389850
Email: enquiries@johncatt.com
Website: www.johncatt.com

© 2020 Craig Barton

All rights reserved.

No part of this publication may be reproduced, stored in a retrieval system, transmitted in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior permission of the publishers.

Opinions expressed in this publication are those of the contributors and are not necessarily those of the publishers or the editors. We cannot accept responsibility for any errors or omissions.

ISBN: 978 1 912906 34 5

Set and designed by John Catt Educational Limited

CONTENTS

Foreword	9
Glossary	11
Introduction	21
Warm-up	33
Chapter 1: Intelligent Practice	35
1.1. What I used to do	35
1.2. What I do now	55
1.3. A shared vocabulary.....	57
1.4. What do I mean by 'Intelligent Practice'?	58
1.5. The three elements needed to support mathematical thinking.....	59
1.6. What does this look like in the classroom?.....	64
1.7. Model the First Relationship	65
<i>Take a Break #1: Subtracting decimals</i>	76
1.8. The Self-Explanation Effect	80
1.9. Student prompt cards	84
1.10. Writing it down	89
1.11. Silent Practice	93
1.12. Paired Discussion	97
1.13. The 4-2 approach	101
1.14. What I do whilst my students are working on the sequence of questions..	102
1.15. Relationships.....	105
1.16. Discuss Relationships.....	109
1.17. Prompts for Delving Deeper	112
1.18. Differentiation revisited.....	120
1.19. Is structured practice even needed?.....	125
<i>Take a Break #2: Rounding to 1 decimal place</i>	128

Chapter 2: Where does Intelligent Practice fit in? 133

2.1. <i>How I Wish I'd Taught Maths: Two years on</i>	134
2.2. A Learning Episode	135
2.3. What do I mean by 'method'?	138
2.4. Introduction	139
2.5. Atomisation	151
2.6. Example-Problem Pair	170
2.7. Example-Problem Pair: Questions and concerns.....	212
2.8. Fluency Practice.....	224
2.9. Intelligent Practice.....	229
2.10. The Example-Problem-Pair-Practice Cycle.....	230
2.11. Method Selection	231
<i>Take a Break #3: Reading scales – Decimals</i>	241
2.12. The 'why'	244
2.13. The Four Ingredients of Problem Solving.....	246
2.14. The Four Ingredients of Retrieval	263
2.15. Formative Assessment	287
2.16. Other elements in the Learning Episode	295
2.17. Want to know more?	303

Chapter 3: Different features of Intelligent Practice sequences305

3.1. Confronting the 'unusual'	306
3.2. Confronting the 'obvious'	314
3.3. Interleaving high-value concepts	318
3.4. Atomisation	322
3.5. Providing a purpose.....	332
3.6. Fill in the Gaps.....	340
3.7. Student-generated sequences	344
<i>Take a Break #4: Carrying out rotations</i>	347

Chapter 4: Intelligent Practice FAQs	353
4.1. FAQs: Student behaviour	354
<i>Take a Break #5: Sharing in a ratio.....</i>	<i>373</i>
4.2. FAQs: Teacher behaviour.....	376
4.3. FAQs: General concerns.....	385
<i>Take a Break #6: Equation of a tangent to a circle.....</i>	<i>394</i>
Chapter 5: Rule.....	397
5.1. What I used to do.....	397
5.2. What I do now.....	398
5.3. What are Rule sequences?.....	412
5.4. Where do Rule sequences fit in?.....	413
5.5. Assessing understanding.....	414
5.6. Rule: Frequently asked questions	418
Chapter 6: Different uses of Rule	423
6.1. Definitions.....	424
6.2. Decisions	434
6.3. Depth	442
6.4. Recap.....	447
6.5. ‘How’ before ‘why’.....	449
<i>Take a Break #7: Averages and range from a list of data:</i>	
<i>Increase, decrease, same?</i>	<i>455</i>
Chapter 7: Pattern.....	457
7.1. What I used to do.....	457
7.2. ‘With the grain’ versus ‘across the grain’	459
7.3. What I do now	461
7.4. What are Pattern sequences?.....	468
7.5. Where do Pattern sequences fit in?.....	468
7.6. Patterns and Structured Variation Grids	471
7.7. Pattern purpose 1: To establish new ideas.....	474

7.8. Pattern purpose 2: To consider Boundary Examples.....	478
7.9. A potential Pattern problem	481
<i>Take a Break #8: Constructions</i>	484
Chapter 8: Demonstration	487
8.1. What I used to do.....	487
8.2. What I do now.....	490
8.3. Straight line graphs.....	492
8.4. Angle at the centre.....	503
8.5. Area of a triangle	512
8.6. What are Demonstration sequences?	514
8.7. Where do Demonstration sequences fit in?.....	515
8.8. Whole class versus individual.....	515
Chapter 9: Summary table	517
Chapter 10: Writing your own sequences	519
10.1. Why reinvent the wheel?	519
10.2. Why write sequences of questions and examples?	522
10.3. A structure for collaboration?.....	523
10.4. Tips for making this process work.....	527
10.5. Tips for writing sequences of questions and examples.....	528
10.6. Sharing is caring	531
Chapter 11: Making this work	533
Conclusion	535
Acknowledgements	545
References	549

For Isaac

Since you arrived I've never slept less, or been happier

FOREWORD

By Jo Morgan

It's impossible not to love Craig's writing. It's also impossible not to feel frustrated when learning the backstory to this book.

Craig's first book was celebrated by teachers for being both enjoyable to read and full of remarkably sensible ideas and honest reflections. Almost immediately it had an astounding impact on what was happening in classrooms up and down the country. New and experienced teachers alike were both challenged and inspired by Craig's ideas and resources. They went into their classrooms and tried new things, reflected and fed back to Craig on what worked well and what didn't. It was a delight to behold a much-needed and relatively widespread shift in maths pedagogy and task design. Of course, I fully expected some of Craig's ideas to be controversial amongst some groups of teachers – after all, as a profession, we are a very long way from agreeing on even the fundamentals of how we should teach maths. But I was surprised when Craig was confronted with vitriol (on social media, by email and in person), which was as aggressive as it was unhelpful. In hindsight perhaps we can laugh about it, but really the sniping nature of debate in education doesn't sit well with any of us.

In this book Craig has taken the opportunity to set out his counter-arguments beautifully. He carefully explains his thinking concerning the practice and examples we give our students, showing exactly what this would look like in the classroom – right down to the words teachers say and the support prompts they provide students. Where his opponents say that they don't have time to explain their ideas, Craig has taken the time.

Right from the first page, Craig's expert storytelling wins us over with its humour and honesty. His anecdotes make us laugh out loud and his explanations are detailed, thorough and always crystal clear. He draws on research and regularly refers to expert opinions gathered through his reading and his podcast interviews, helping to make those opinions accessible to teachers who otherwise might not be exposed to these important ideas. He also speaks from experience, not only from his own teaching but also from real teachers in real classrooms, showing us how they have adopted and adapted the ideas from his first book.

It's really important to follow Craig's instructions throughout this book to get the most out of it. If he tells you to do a task, pick up a pen and do it. If he asks you to reflect on it, put your pen down and have a think. It's not until you try the tasks that you understand what they are designed to achieve. A cursory glance is simply not enough; skim-reading doesn't work here. When I've taken the time to complete Craig's tasks I've always been surprised when what looks like a boring worksheet ends up making me think deeply and notice things that I have never noticed before. At one point I was completing an exercise in this book and was totally taken aback by a result that I wasn't expecting. I was delighted to find myself mathematically stumped and thinking hard. There is much to learn from this book, but doing so takes time. There are no quick wins.

When Craig describes assessing students using 'thumbs up' as being 'like a roomful of Caesars passing judgements on a defeated gladiator', I really laughed. In fact, I laughed throughout the book. Craig has an incredible talent for writing about maths teaching in a thoroughly entertaining way that all teachers can relate to. It's not just the refreshing honesty and amusing anecdotes that make this book another must-read for maths teachers. It's the carefully thought-out and researched ideas, and the intricately explained pedagogy that makes it essential reading. Craig challenges us to try new things and gives us everything we need to go into our classrooms and do so.

This is another game-changer of a book from the formidable Mr Barton. As he strives to be a better teacher, he carries us all along with him.

A triumphant second album.

*Jo Morgan is a maths teacher, the creator of resourceaholic.com and the author of *A Compendium Of Mathematical Methods*, you can find her on Twitter at @mathsjem*

GLOSSARY

This book contains lots of terminology that may mean different things to different people. This can make collaboration and discussion difficult. So, each time I introduce a term in the text I will do my best to explain what I mean by it, supported by an example. Below you will find a list of the key terminology from this book, complete with a definition. It probably won't make that much sense at this stage, but hopefully this page will provide useful to return to if you are unsure what I am going on about.

4-2 approach

The structure I use when students are working through Intelligent Practice sequences:

- 4 minutes of Silent Practice
- 2 minutes of Paired Discussion
- Repeat...

AQA

One of England's exam boards or awarding bodies.

Atomisation

The breaking down of a complex method into different sub-steps so that students can better understand how they fit together.

Atom

An individual sub-step required for a complex process

Attending

Students concentrating and focusing on a critical feature of a worked example, problem or sequence of questions or examples.

Boundary Example

An example that helps distinguish between having and not having a specified property.

Check

Stage 3 of the mathematical behaviour at the heart of this book. Students check the answer using the method they have been taught, via Paired Discussion, or when the teacher reveals the answer.

Choral Response

A teaching technique where all students recite the answer to a question at once.

Confronting the 'obvious'

A feature of Intelligent Practice sequences where students are exposed to results that may be obvious to teachers but surprising to students.

Confronting the 'unusual'

A feature of Intelligent Practice sequences where students are exposed to results and examples early that are at the boundary of a concept.

Curse of knowledge

A cognitive bias that occurs when an individual, communicating with other individuals, unknowingly assumes that the others have the background to understand.

Diagnostic Questions

My preferred form of Formative Assessment. Diagnostic Questions are multiple choice questions consisting of one correct answer and three wrong answers, with each incorrect answer designed to reveal a specific misconception. I discuss Diagnostic Questions in detail in Chapter 11 of *How I Wish I'd Taught Maths*, and you can access tens of thousands of questions for free by visiting my website: **diagnosticquestions.com**

Differentiation

Ensuring as many students as possible are given the appropriate level of support and challenge to meet their individual needs.

Discuss Relationships

This is where I bring the whole class' attention to the board after they have worked through a sequence of questions or examples. Here I reveal the answers one at a time and choose some key relationships to discuss.

Example-Problem Pair

My five stage process for delivering worked examples. It is designed to help my students be actively involved and attend to critical aspects.

Example-Problem-Pair-Practice Cycle

The phase of the Learning Episode where I cycle between worked examples and practice. The practice may take the form of Intelligent Practice or Fluency Practice.

Expect

Stage 2 of the mathematical behaviour at the heart of this book. Based on their reflection and the answer to the previous question, can students form any expectations about this answer?

- Will the answer increase/decrease/stay the same?
- Do you expect a specific value for the answer?
- Will the answer look different?
- Will the way you work out the answer change?
- Can you articulate why?

Explain

Stage 4 of the mathematical behaviour at the heart of this book.

- If a student could not form an expectation before, can they explain the relationship now?
- If they are surprised, can they now explain why?
- If they are not surprised, can they think of a way of helping someone who doesn't understand it yet?

Expertise Reversal Effect

The Expertise Reversal Effect refers to the reversal of the effectiveness of instructional techniques on learners with differing levels of prior knowledge. Specifically, as students become more knowledgeable in a given domain, less teacher-directed instruction is most suitable.

Fill in the Gaps

A type of Intelligent Practice sequence where students can work across or down to fill in the gaps in a grid.

Fluency

We consider someone to be fluent in a technique, skill, idea, concept or facts at the point at which they no longer need to give attention.

Fluency Practice

The stage of the Learning Episode that precedes Intelligent Practice. Questions are unrelated so a student's attention can be focused on the method and not on the relationships between questions.

Formative Assessment

The process used to recognise and respond to student learning in order to enhance that learning, during the learning.

Four Ingredients of Problem Solving

They are:

- SSDD Problems
- Goal-Free Problems
- Purposeful Practice
- UKMT Questions

Four Ingredients of Retrieval

They are:

- Starters
- Low-Stakes Quizzes
- Mixed-Topic Homeworks
- Interweaving

Frayer Diagram

A structure for students to demonstrate what they know about a concept or idea. You can find out more via **frayer-model.co.uk**

Gap of understanding

A feature of Pattern sequences where the pattern skips out several steps. Students are compelled to think hard about the question and not just continue the pattern.

GCSE

The series of national exams that most students in England sit across all the subjects they study at the end of Year 11 (when they are 15 and 16 years old). Some students may take their GCSEs earlier, and some may need to re-sit them later.

Goal-Free Problems

One of my Four Ingredients of Problem Solving. These are multi-step problems where the end goal has been removed. You can access these via **goalfreeproblems.blogspot.com**

High-value concepts

The areas of mathematics that form the foundations to lots of other mathematical ideas.

Hypercorrection Effect

Occurs when students have given a response to a question which they feel confident is correct, but then discover that their response was in fact wrong. The feeling of surprise leads them to better correct the original problem. Moreover, they are more likely to remember the correction in future, thus improving long-term learning of the idea or concept.

Illusion of collective wisdom

Making inferences about the understanding of the individuals in a pair or a group based on their collaborative output is problematic as we do not know if the students could have got there on their own.

Illusion of retrieval

When students do not put in the necessary individual effort when tackling questions based on ideas they have met in the past, and therefore do not benefit as much as they could from the power of retrieval.

Intelligent Practice

Sequences of questions which enable students to gain practice in carrying out a mathematical method, whilst at the same time providing opportunities to think mathematically.

Interleaving Effect

Contrasts a 'blocking' approach, whereby students study the same type of material over and over again before moving on to a different type of material, against an approach where students practice all of the problems in an order that is more random and less predictable. The latter approach has been found to enhance learning and transfer.

Interweaving

Integrating different areas of mathematics into the topic currently being studied.

Learning Episode

The amount of time required to grasp a novel idea.

Low-Stakes Quizzes

One of my four ingredients of retrieval. These are mixed-topic quizzes that students mark themselves.

Mathematical thinking

In *Thinking Mathematically*, John Mason, Leone Burton and Kaye Stacey examined the basic components of mathematical thinking. They identified four stages:

- Specialising: Trying special cases, looking at examples.
- Generalising: Looking for patterns and relationships.
- Conjecturing: Predicting relationships and results.
- Convincing: Finding and communicating reasons why something is true.

Method

The steps required to solve a problem.

Method Selection

The phase of the Learning Episode focused on developing students' proficiency in selecting an appropriate method to solve a problem.

Mixed-Topic Homeworks

One of my four ingredients of retrieval. These are homeworks that are not focused on a single mathematical idea but instead feature ideas students have met at various points in the past.

Modality Principle

People learn more deeply from graphics and narration than from graphics and on-screen text.

Myth of copying it down

The belief – either explicit or implicit – that just because students copy something down, they understand it and will remember it.

Narration

Stage 2 of my Example-Problem Pair process. The teacher asks questions about critical steps in the worked example.

Paired Discussion

Students engage in positive mathematical discussions with their partner. Prompts are available to help support these discussions.

Problem Solving

Applying a method – or selection of methods – to a novel situation.

Prompts for Delving Deeper

These are challenges I give to students either as a whole class following the Discuss Relationships phase, or to small groups who finish a sequence early. They are designed to further promote mathematical thinking.

Purposeful Practice

One of my four ingredients of Problem Solving. They are activities that provide practice of a key method, whilst at the same time providing opportunities for students to think deeper. You can access my Venn diagram activities (which are one form of Purposeful Practice) via **mathsvenns.com**

Read the Maths

Stage 3 of my Example-Problem Pair process. Students study the worked example from start to finish, annotating where needed.

Reflect

Stage 1 of the mathematical behaviour at the heart of this book. Students ask themselves what has changed and what has stayed the same between two questions or examples.

Redundancy Effect

The Redundancy Effect is the finding that redundant information causes extraneous load making thinking harder. Redundant information is hard to ignore, therefore it must be processed. Even filtering out is effortful. This takes up valuable space in working memory.

Relationship

The connection that exist in sequences and examples between two questions or two answers.

Retrieval

The effortful act of drawing upon knowledge stored in long-term memory.

Self-Explanation Effect

The finding that self-explaining helps the learner integrate new knowledge with existing knowledge, refine existing mental models, and can improve the acquisition of problem solving skills.

Self-explaining

The internal process where students are able to fill in missing information, monitor understanding, and modify fusions of new information with prior knowledge when discrepancies or deficiencies are detected.

Silent Practice

Students work through sequences of questions without talking to their partner or without interference from their teacher.

Show Call

Stage 5 of my Example-Problem Pair process. Students' work is shown to the rest of the class to identify misconceptions and showcase excellence.

Silent Teacher

Stage 1 of my Example-Problem Pair process. The teacher models the worked example in silence.

Spacing Effect

Learning is better when two or more exposures to information are separated in time (*i.e.* spaced apart) than when the same number of exposures occurs back-to-back in immediate succession.

Split-Attention Effect

The Split-Attention Effect occurs when students must split their attention between multiple sources of information, which results in extraneous cognitive load

SSDD Problems

One of my Four Ingredients of Problem Solving. Standing for *Same Surface, Different Deep*, these consist of four questions which share a common characteristic (image, context, number, expression and so on), but whose deep structure are different. You can access these via **ssddproblems.com**

Starters

One of my four ingredients of retrieval. Starters often happen at the start of the lesson (*surprise, surprise*) and are a means of ensuring each lesson consists of a selection of topics students have studied at various points in the past.

Student-generated sequences

Sequences of Intelligent Practice questions written by students.

Testing Effect

For Bjork and colleagues, retrieval is not a neutral event. Each time retrieval happens, both the storage and retrieval strength of that memory increases. Thus retrieval better secures that knowledge in long-term memory, slows down the rate of forgetting, and makes it more likely that students will be able to access that knowledge in the future. This is also known as the ‘Retrieval Effect’.

Think, rehearse, share

My process for asking a question, giving students time to think on their own in silence, then rehearsing their answer with their partner (and listening to their answer), before sharing it with the class.

UKMT Questions

One of my Four Ingredients of Problem Solving. These are questions written by the United Kingdom Mathematics Trust (UKMT) which often present topics in interesting ways. You can find a selection of questions sorted into topics at **diagnosticquestions.com/UKMT**

Variation Theory

Based on the work of Frances Marton, Variation Theory posits that discernment of a critical aspect of a phenomenon results from experiencing variation that correspond to that aspect.

Worked example

The teacher models how to carry out a method or solve a problem.

Year 1, 2, 3, 4 and so on

This is the schooling system used in England. In most cases, students in these year groups will be:

- Year 1: 4 and 5 year olds
- Year 6: 10 and 11 year olds (final year of primary school)
- Year 7: 11 and 12 year olds (first year of secondary school)
- Year 11: 15 and 16 year olds (students sit their GCSE exams)
- Year 13: 17 and 18 year olds (students sit their A Level exams)

Your Turn

This is where students have an opportunity to see what they have learnt. Your turn occurs:

- following a worked example where students try a related problem.
- following a Rule sequence of examples where students attempt a selection of questions.

You can find the links to the questions, photos, diagrams and activities featured throughout this book at: **www.mrbartonmaths.com/booklinks**

INTRODUCTION

The difficult second album

There's a concept in musical circles known as the difficult second album.

A band or artist spends their whole life gathering the experiences, ideas and heartaches that provide ample material for their first offering. They release the album, and – if it is successful – spend the next year touring it, taking a break from the gritty reality of their previous lives, living out a cushy existence far removed from what they had known. When they eventually settle down to work on their second album they find they are out of touch with reality. Worse still, they simply have nothing left to say. They plough on regardless, bleeding past glories dry, convincing themselves that the end result is a masterpiece. But everyone knows it is a pale imitation of what came before. A crushing disappointment for all concerned.

It's quite sad, really.

Anyway, my name is Craig Barton, I am a maths teacher and the author of the bestselling *How I Wish I'd Taught Maths*. I'm currently on a lovely sabbatical from my school in Bolton, touring the UK and overseas with the ideas from that book, dining out on Premier Inn meal deals and free hotel biscuits. This is my second book. It's based on an idea from the first one. And I think it's great.

From legend...

If any of you are still reading, let me tell you a little bit about why I have written *Reflect, Expect, Check, Explain*.

In January 2018, *How I Wish I'd Taught Maths* was released. I was incredibly nervous about its reception. I remember checking Twitter about six minutes after the book had gone on sale, finding no mention of it whatsoever, and having a breakdown in front of my wife, sobbing, '*they hate it, they hate it*'.

I was subsequently blown away to find that a tale of 12 years of maths teaching ineptitude, and my attempts to right those wrongs by using evidence-informed strategies, seemed to have resonated with many teachers. These teachers were UK based and international, experienced and novice, young and old, primary and secondary. Teachers of other subjects seemed to be able to relate to my experiences too.

After basking in the glory for a few months – and checking my Amazon sales rank approximately every 43 seconds – I set about completing two projects based on ideas I described in my book.

The first was to further develop the idea of SSDD Problems.

For those of you who have not read *How I Wish I'd Taught Maths* (still available from all good and evil book stores), SSDD stands for Same Surface, Different Deep (and not same sh*t, different day, although that confusion may explain the number of hits the website gets). SSDD Problems consist of four questions that look similar on first inspection – perhaps they have the same central image, or share common words or context – but which each require a different area of mathematical knowledge to solve them. We will discuss exactly where SSDD Problems fit into my teaching in Chapter 2 of this book.

I wrote an initial batch of around 50 SSDD Problems spanning a whole host of maths topic areas. I showcased these at various conferences I spoke at and schools I visited. They seemed to go down well. I said to the teachers I worked with, 'you write me one set of SSDD Problems, and I'll give you 50'. Whilst this was happening, I was secretly building a website to house these SSDD Problems. On Saturday the 10th of March, 2018, during my workshop at the La Salle MathsConf in Kettering, I unveiled my creation: **ssddproblems.com**.

I thought my workshop had gone well. But when I eventually made it back up the treacherous M6, fixed myself a cup of Mellow Birds, sat down on my settee, and cracked open Twitter, I was amazed. People were loving the website. I was described as a hero and a legend, and the maths profession was lucky to have me. Needless to say this went straight to my head, and my repeated attempts to remind my wife how lucky she was to be married to such a legend did not go down all that well.

In the following weeks teachers from all around the world both visited and shared their creations on **ssddproblems.com**. Within two months the website had been visited over 200,000 times and housed 300 sets of problems. I could not have been happier, and this spurred me on to start the second project from my first book.

... to dangerous and clueless

The concept of minimally different problems, and what I termed Intelligent Practice, formed a relatively small part of *How I Wish I'd Taught Maths* but, for most of 2018, it became my obsession. When trying them out with my Year 11s (in the build up to the summer GCSE exams, I saw the power of these sequences of questions in terms of the connections they helped my students make, the depth of thought they required, and the discussions they provoked.

So, once again, behind the scenes, I began creating a bank of these sequences of questions, bringing fellow Intelligent Practice lovers Jess Prior (@FortyNineCubed) and Ben Gordon (@mathsmrgordon) on board to help with my project. I built the website **variationtheory.com** – another community where teachers from around the world could share, collaborate and learn from each other.

We set our sights on a big launch: the 23rd of June, 2018, at MathsConf in Manchester. Yes, my very own North West of England was about to bear witness to one of the most significant releases in the history of maths education. I was picturing standing ovations, knighthoods, statues and women begging me to father their children. ‘Barton has done it again’, the headlines would read. This was going to be the best thing ever.

What could possibly go wrong?

The workshop itself went well. Jess and Ben were outstanding, and the teachers in the room – one of whom was Bruno Reddy, the man who first showed me the power of minimally different examples many years before – seemed to like the website. So once again I drove home, made myself a Mellow Birds, sat down on my settee, and opened up Twitter. I could not believe my eyes. I thought I had logged into the wrong account.

The first tweet that caught my attention was: ‘Is this really what maths education has been reduced to?’ Quickly followed by: ‘Looks like a website of boring worksheets to me.’ And my personal favourite: ‘Craig Barton – the most dangerous and clueless man in education.’ (This tweet was subsequently deleted.)

I have never been described as dangerous before, so I was quite happy with that one. Clueless, not so much.

Sure, many teachers seemed to like the website. But there were a significant few who did not. And they made sure that I knew about it. In the following weeks and months, blog posts were written, tweets were tweeted, and emails and private messages arrived from people who I had (and still have) a tremendous amount of respect for. They all said the same thing: this was not good, not good at all.

I will be honest with you, the reaction really upset me. I was upset for Jess and Ben, who had put in hours and hours of work writing sequences of questions for the site with their only motivation being a desire to share good practice. But I was also upset for me. Not just because I had put in many hours as well, but because I had seemingly misjudged the whole concept. Something that I thought was really important for student understanding, others thought was rubbish, harmful in fact.

It really knocked me, far more than it should have done.

Taking a long, hard look at myself

As I considered the criticism in more detail, read as much as I could about the principles of Variation Theory and how they might apply to Intelligent Practice, worked with teachers, tried sequences of questions and examples with students, tweaked or scrapped them accordingly, and wrote more sequences, two things happened.

First, I became more aware of the power of this approach than ever before. Not only could it be used to enable students to practise key methods, but it could also be used to differentiate, provoke discussion, promote mathematical communication, encourage conjecturing, stimulate creativity, reveal different representations, revise for exams, interleave high value skills, atomise complex concepts, and help students make crucial decisions. In other words, they could help students think mathematically. This revelation developed when I was lucky enough to try these sequences out with my own students in sunny Bolton and those all around the country.

Second, I noticed there were two groups of practitioners who did not like what I was doing. There were those who thoughtfully listened to what I was trying to achieve, and concluded that the approach was wrong. They did not believe that such structured practice was needed, and thought that the approach I was advocating was no better than a random worksheet. We have different views on how students best learn mathematics, and (on a good day!) their critique of my approach is a positive force that helps me think hard about what I believe. I completely respect their opinion, but I disagree with it. I address this viewpoint at the end of Chapter 1 of this book.

Additionally, there also seemed to be a significant number of practitioners who – through no fault of their own – did not understand what I was saying.

At the start of workshops I would ask, ‘what does Variation Theory mean to you?’ or ‘have you used minimally different examples?’ and I could see eyes rolling. For some teachers those ideas seemed obvious – old ideas in new packaging. To others they represented yet another fad they had heard about on some course or seen mentioned on Twitter. Some had tried out a sequence of two with their own students and found the results uninspiring.

However, when I was lucky enough to share the specifics of my approach with these teachers – especially when they tried the sequences out with their students accompanied by the supporting mathematical behaviour I advocate – things started to change. Teachers messaged to say they were surprised how well their

lesson had gone. They shared comments from their students who seemed to be grasping the previously ungraspable. They sent insightful questions students had asked about the sequences. I started to think that perhaps some of the issue was my poor communication and not the concept itself.

I needed to do something about this.

And so we have the book that you are (hopefully) about to read. It is a book I thought long and hard about whether I should write as I know it will be divisive. It is a book I have deleted three times (thank goodness for Dropbox's automated back-ups), and abandoned many more.

There is nothing new in here. These ideas have been around for centuries. And whilst I speak and write about them like a boy who has experienced his first Christmas, I know some readers may find themselves saying 'yes, and tell me something I don't already know', but I taught for 12 years without knowing them. More than that, this book is my attempt to take these ideas further. Specifically, I aim to develop a structure to give these powerful ideas the best chance of working with the majority of students, in the majority of classrooms, for the majority of mathematical ideas.

The intended audience

In the introduction to *How I Wish I'd Taught Maths*, I made it clear that the book was written by a secondary school maths teacher for secondary school maths teachers. I genuinely had no idea if the ideas I described had any useful applications outside of my secondary maths bubble.

I want to offer the same word of warning for this book, because this book is niche, to say the least.

Every single example is from the domain of mathematics. I have no idea if the specific application of Variation Theory that I describe in this book can be applied to other subjects. My instinct, backed-up by conversations with colleagues, is that there may be opportunities in science, computing and possibly modern foreign languages. However, I cannot say so with any certainty.

Likewise, I can only speak from the experience of teaching secondary school mathematics. When possible I visit primary schools, and am constantly amazed by the depth of thought, difficulty of mathematics, and high standard of teaching that I see. I certainly learn more from those visits than the primary teachers do from me. And whilst I have had the opportunity to try some of the activities described in this book (and on my **variationtheory.com** website) with students as young as seven years old, I make no claim that these ideas will transfer.

That said, I have purposefully included examples from both primary and secondary mathematics, and from as many different areas of the mathematics curriculum as possible, to try to convey the breadth of application of these principles.

How to get the most out of this book

Reading can be a passive experience. I have built two features into this book that I hope will help you – should you wish to partake – get as much out of it as possible:

1. What do you think?
2. You do the math(s).

1. What do you think?

Any time I am lucky enough to work with teachers, whether in their school or during a workshop, I always start by saying the same thing: don't believe a word that I say, assume I am talking nonsense, and therefore question everything. At this stage the organiser of the day is usually looking a bit worried. I explain that my role is not to say 'do this' or 'don't do that'. It is not to promise that something will definitely work. It is not even to give advice. My role is to present what I used to do, why I changed my mind, what I do now, why I do it, and how I evaluate the impact it has had.

That gives the teachers I work with a very important job: to think hard about the ideas I present and ask themselves first, 'Do I think this idea has any worth?' and, if so, 'What would I need to change to make this work for my students?'. The most carefully designed teaching, backed by the most robust empirical evidence, will fall flat on its face if students do not put effort in. Teaching is a relationship business, and no-one knows your students better than you, or has the potential to teach them as well as you do.

The same is true of the ideas presented in this book.

I will be describing my views on carefully varied sequences of questions and examples, and how I have tried to help my students develop the behaviour of *reflect, expect, check, explain* to get the most out of these sequences. I will show you exactly what it looks like in my classroom, explain where this fits into my teaching, and how it interacts with other key ideas. I will also share how schools around the world have taken and adapted these ideas, but it will be down to you – if you think the ideas have any worth – to try to make them work for your students.

With this in mind, at key points I will ask you to pause and think about something. This will be indicated by a box entitled 'What do you think?'. You will be asked

if you can relate to what I describe, what you agree with, what seems wrong and, crucially, what you would need to change to make this work in your context.

Answers to these questions will likely take time to discover. They may involve conversations with colleagues, micro-experiments with students, plenty of failures, and many periods of reflection, but I believe the answers will come.

I know lots of teachers read my first book together as a department, discussing a chapter at a time in departmental meetings. If appropriate, these questions could provide the stimulus for these discussions.

2. You do the math(s)

If I look back at some of the many woeful lessons I have taught over the years and consider exactly why they went wrong, one feature often emerges: I did not do the examples, questions, or activities myself first before giving them to my students. I thought a casual glance would suffice.

It is only when we sit down, pen and paper in hand, and go through the mathematics as we expect our students to do that we are able to spot difficulties, prerequisite knowledge, interesting quirks, possible points of discussion and unintended ambiguities. This is perhaps our best chance to overcome the dreaded *curse of knowledge*¹ that so often prevents teachers from anticipating how students will fare in the work we give them, and thus from being adequately prepared to help them.

With that in mind I have packed this book full of maths. Wherever possible I would love you to do the sequences of questions and examples that I present, not just as a teacher but as a mathematician. There will be the temptation to do the questions in your head, but please try to resist! Grab a pencil and a piece of paper. There will also be the temptation to skip out the odd question, but be strong. The questions and examples in this book are parts of sequences, and as such are designed to be viewed as a whole. When you have completed a sequence, please reflect on the prompts following the sequences of questions and examples.

For maximum effect, get a colleague – or your entire department – to do the same. Then discuss your respective answers to the prompts.

What do you agree upon?

Where do you differ?

Can you persuade or be persuaded?

What would you need to change for this to work with your students?

1. For more about the dangers of the *curse of knowledge*, see www.bit.ly/34z4e79

Then try them with your students.

What worked?

What didn't?

What did you learn?

What will you do different next time?

If you repeat this process for as many of the sequences of questions and examples in this book as possible, then hopefully you will get a lot out of it.

I also want to make it clear that I do not think any of the sequences in this book are perfect, or the finished product. Far from it. They are all works in progress. I view them as a springboard to promote thought, discussion and debate. They can all be improved. This may involve tweaking them to suit the needs of your class, or tearing them apart and starting again. You may spot how to do this when working through them yourself, when discussing with a colleague, or when you try them for the first time with your students. Good sequences of questions and examples evolve over time via the collective wisdom of all who engage in them.

All the sequences in this book – along with over 500 more – are available for free at my website variationtheory.com in editable form. There is also opportunity to suggest your own changes, and I warmly encourage you to do this.

Calculating an estimate for the mean

Talking of doing the maths...

Chapters 1 and 2 are based around a sequence of questions concerning calculating an estimate of the mean from grouped data.

This choice is deliberate. It is a dull topic, or at least it was the way I would teach it. As I will describe, the practice I gave my students was very procedural – they calculated midpoints, multiplied numbers and summed totals like robots without the need to think deeper. I want to show how I have changed this practice (Chapter 1) and where it fits into a sequence of lessons (Chapter 2), so I can use the remainder of the book to apply these principles to other mathematical ideas.

Obviously, calculating an estimate of the mean from grouped data is not taught in primary schools, but I am also aware from conversations with colleagues that this is a rather alien concept in the likes of Scotland and Canada. If this idea is not familiar to you, you are in fact at an advantage because your experiences may be more in line with those of our novice students. However, some knowledge of the method is useful so you can partake in the sequence of examples in Chapter 1.

Therefore, should you wish to brush up on your calculating an estimate of the mean skills, here is a typical question:

Worked example

Calculate an estimate of the mean weight.
Give your answer to 1 decimal place.

weight w (g)	Frequency
$0 \leq w < 10$	5
$10 \leq w < 20$	8
$20 \leq w < 25$	10
$25 \leq w < 40$	4

Figure 1 A typical question on calculating an estimate for the mean

And here is a possible method to solve it:

Worked example

Calculate an estimate of the mean weight.
Give your answer to 1 decimal place.

weight w (g)	Frequency	Midpoint	Midpoint $\times$ Frequency
$0 \leq w < 10$	5	5	25
$10 \leq w < 20$	8	15	120
$20 \leq w < 25$	10	22.5	225
$25 \leq w < 40$	4	32.5	130
TOTALS	27		500

$25 + 40 = 65$
 $65 \div 2 = 32.5$

Mean = $500 \div 27$
= 18.518...
= 18.5g (1dp)

Steps

1. Add a Midpoint column and calculate the midpoint of each group

2. Add a Midpoint $\times$ Frequency column and work out the product of each group's midpoint and frequency

3. Add a Totals row at the bottom

4. Total up the Midpoint $\times$ Frequency column and the Frequency column

5. Divide the total of the Midpoint $\times$ Frequency column by the total of the Frequency column

6. Round your answer to the required degree of accuracy

Figure 2 A worked example on calculating an estimate for the mean

And just to check your understanding, have a go at this Diagnostic Question, written by an English exam board, AQA:

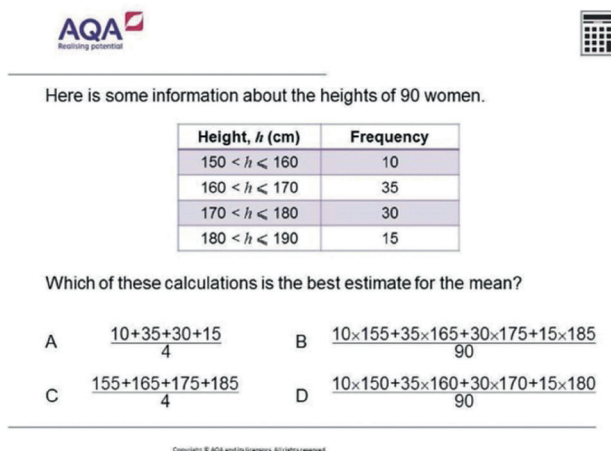


Figure 3 A Diagnostic Question from AQA

This question was written by AQA for the Diagnostic Questions website. You can find all of AQA's questions here: diagnosticquestions.com/AQA

Answer at the end of the introduction!

Do I need to have read *How I Wish I'd Taught Maths* before I read this book?

The marketer in me wants to say 'yes, definitely, and buy one for your granny too', but this is designed to be a standalone book. It takes one relatively small idea from my first book and delves deep into it, assuming no prior knowledge.

Throughout this book I will reference ideas from *How I Wish I'd Taught Maths*. This is particularly the case in the 40,000+ word epic that is Chapter 2, where I look at how Intelligent Practice fits into my model of teaching. Here I take the opportunity to revisit the core ideas from my first book, examining how they have developed in the two years since the book's release.

So, if you did buy *How I Wish I'd Taught Maths*, and you have not subsequently flogged it on eBay for 53p, then you might like to have your copy to hand to look up the genesis of those ideas and the research they were based on, but it should work just fine on its own.

Where can I access the resources?

The maths and the diagrams

This book is rammed full of maths. There are an abundance of sequences of questions and examples, as well as sets of SSDD Questions, Diagnostic Questions, hooks and beautiful activities written by the finest task creators of them all. There are also more than a couple of flow diagrams that I hope will help illustrate my approaches.

I hope after having a go at the activities and ideas in the book, you will want to try them out in your classrooms. Instead of providing far-from-catchy URLs for each one, I have created a single webpage where everything is linked to. You can find the page here: mrbartonmaths.com/booklinks

Research

Research papers are referenced throughout the text, and their full details can be found in the References section at the end of the book. You will find many of the original papers available to read as PDFs in my research collection here: mrbartonmaths.com/teachers/research

Without further ado...

This is very much the difficult second album. I am even more nervous than when I put out *How I Wish I'd Taught Maths*. I know people will not like what I have to say here. For some it is obvious, for others it is poor pedagogy.

But I believe in these ideas.

I know there will be bad reviews posted, dismissive tweets tweeted and challenging blog posts written. And I know they will upset me. But I relish these challenges as they make me think more. The aim of this book is not to convince you that I am right. I wrote it to clarify my own thoughts, and to provide a justification and structure for those who wish to give the ideas a try. I aim to convey as clearly and comprehensively as possible exactly what it is I am advocating, why I believe in it, and what it looks like in the classroom. That way, if you do decide you disagree, you at least know *what* you are disagreeing with.

Okay, that's enough build up. We best get cracking. Thank you for taking a chance on this book. I really hope you enjoy it.

– Craig ‘dangerous and clueless’ Barton

P.S. The correct answer to the Diagnostic Question is B.