A COMPENDIUM OF

MATHEMATICAL METHODS

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Preface

This compendium presents a selection of mathematical methods for topics in the secondary maths curriculum.

Some of the methods featured are used widely in schools around the world; others are only used in a small number of countries. Some have been in use for generations and others have fallen out of fashion.

I've included a number of procedures in this book that I wouldn't use or teach. I've chosen not to be overly critical of them here, apart from the occasional caveat. There are few (if any) longitudinal studies providing evidence that one approach is superior to another. It is left to you to determine which are the 'best methods' – which ones to use yourself when doing mathematics, and which ones to teach your pupils. In making these important decisions, there are a number of factors to consider:

Do you understand why the method works?	Is the method easy to explain? Will pupils understand how it works?	Is the procedure easy to remember? Multi-step algorithms will probably need a lot more practice than more intuitive methods.
Does the method lead to common misconceptions? How will you deal with that?	Is the method efficient? Is it elegant? Is it intuitive? Would you use it yourself?	Does your chosen method hold for maths you will teach in the future? Will it need to be replaced when the problems become more complex?
Do you want to select methods for your pupils, or allow them to choose? If you intend to give them some choice, do you want this decision being taken at a whole-class level or an individual level?	Do you want consistency of methods within your school? Does it matter? Consider what happens if pupils change teachers.	Do your pupils use an online homework platform? Do they use textbooks? Are your chosen methods consistent with the methods pupils will see elsewhere? Does it matter?

This book features some extracts from antique maths textbooks. These are provided purely for your interest, not because I think the Victorians got everything right in maths education! It can be rather insightful to see how procedures were described in the past and what exercises used to look like.

It's important that you read this book with pen and paper to hand so you can have a go at each method. It can be very satisfying to do familiar maths using a new method, plus it will help you make sense of what's going on. In each chapter a few questions are provided for this purpose. Note that a few questions might not be sufficient – normally a lot more practice is required to really get the hang of a new method and to start to evaluate its pros and cons. In fact sometimes teaching it is the best way to really understand and evaluate it.

When reading this book, bear in mind that most methods differ from alternatives only in layout. As you'd expect, it is simply the surface structures and processes that differ – the underlying mathematics is identical. As a result, I haven't covered every possible method in this book. Far from it! Some methods are only subtly different from others and I'm unable to include every single version. However, if I've missed something that is fundamentally distinct then please do contact me for possible inclusion in future editions.

One final but important point – you must talk to your colleagues about methods! Find out how they do things; find out what they teach. This kind of subject knowledge development is vital for any maths department. But don't be surprised if you meet scepticism about different approaches, and reluctance to try new things. Some teachers are defensive about their subject knowledge and will say, 'I've always taught my pupils this way and they get all the questions right in the lesson. Why do I need to know another way?' Bear in mind that when someone tells you that they use the 'best method', it may be that they have simply never tried any others. Some of the very best maths teachers are those who take the time to research their subject in greater depth. Exploring new methods can help us make sense of things, even if we choose not to teach those methods. All I ask is that you read this book with an open mind and willingness to learn.

Contents

Frequently Asked Questions

1. Won't my pupils get confused if I show them lots of different methods?

You don't have to show them every method in this book! That might be a step too far. But I believe there are benefits to exploring different methods with our pupils. Teachers in some high-performing jurisdictions do this. It can be quite revealing and may lead to a deeper level of understanding. But you need to get the timing and approach right to ensure that alternative methods don't do more harm than good. Once a pupil has developed procedural fluency and an initial comprehension of a chosen method, exploring alternative approaches is likely to deepen their understanding.

2. I have a method that works well. Why should I care about other methods?

There are many reasons why teachers should take an interest in alternative methods.

- It generates rich mathematical discussion – there's not widespread agreement on what methods are 'best'. In my experience teachers can be surprisingly defensive about the methods they use.
- This is what 'subject knowledge development' looks like for experienced maths teachers. Teachers should be curious mathematicians – it's fun to explore methods and learn new things.
- Teachers should be able to offer a toolkit of approaches, not just their 'favourite' method. Plus remember the 'each to his own' principle: your pupils and your colleagues may have different preferences to you.

3. I'm a head of maths – should I tell my teachers what methods they must teach?

Consistency across a maths department is helpful, particularly if pupils have a number of different maths teachers throughout their time at the school. One common approach is to insist that all pupils are exposed to one particular method, but teachers are allowed to teach alternatives. For example a head of maths might say that every student in the school must be shown the grid method for expanding polynomials at some point, but that teachers are welcome to teach alternative layouts too and are not obliged to use a grid as their primary method in explanations and examples.

4. If my pupil uses an 'unusual' method, will they still get method marks in public exams?

In England, unless the question states otherwise, any valid mathematical method is eligible for method marks in GCSE and A level exams. Public exams are typically marked by experienced maths teachers who should be adept at recognising and following unconventional approaches. If they see an incorrect answer following on from a method they do not recognise then they may seek a second opinion from a superior. In short, restrictions are not placed on methodology.

Unfortunately, at the time of writing, the same cannot be said for the public exams sat by pupils in Year 6 in England. If a child gets a multiplication or division question wrong then only 'formal' (government-prescribed) methods of multiplication or division will earn method marks. This unprecedented central prescription of mathematical methods is highly controversial.

5. Is there any research on methods?

There's very little, if any, recent research on the relative effectiveness of most of the methods described in this book. Over the years, there has been a fair amount of research conducted on primary school pupils concerning written methods of arithmetic, but these studies normally test short-term accuracy and efficiency rather than understanding, long-term retention and so on. These days, given that calculators are now ubiquitous, efficiency in arithmetic is far less of a concern than it used to be.

It's interesting to note that in the early 1900s, some writers were rather vocal with their opinions on which methods were 'best'. This suggests that this was a period in which there was some controversy over which methods should be taught in schools – perhaps more so than today. For example, P.B. Ballard refers to what he calls 'The King's Highway' in his book *Teaching the Essentials of Arithmetic* from 1928. 'The King's Highway' is his interpretation of 'the best method'.

I don't sit on the fence: I come down definitely on one side or the other. Let me cite a few instances. I think the method of teaching subtraction by decomposition a vicious method. I am convinced that the policy of shirking long division till late in the course and substituting division by factors is wasteful and ineffective. I hold that compound multiplication by factors is a clumsier method than direct multiplication in line, and that the unitary method of working proportion is more cumbersome than the fractional method. And I am a great believer in the King's Highway – in having one good standard method of working a given type of sum. I have observed that those who show too eager a desire to avoid the beaten track and discover short cuts often come to grief. They either lose their way or arrive late. Meanwhile their more pedestrian classmates who have gone the longest way round have really found the nearest way home. We must distinguish (if we can) between new ideas that have come to stay and new ideas that arise from chance and change – ideas which are in the fashion, and have in consequence a certain air of smartness, but come off badly when subjected to the wear and tear of the classroom. The worst of it is, when one idea gets into fashion it pushes another idea out of fashion. And the other is often the better of the two. It was the fate that befell 'equal additions' when 'decomposition' got into vogue; it is a disaster that threatens to befall that good old method of multiplying decimals – counting the decimal places.

Chapter 1
Subtraction

For most topics in school mathematics, there has been little research into the 'best' methods. But mathematicians have spent centuries fiercely debating the best way to subtract. Most of us were taught one method at a young age, oblivious to the fact that for many years there were two methods that dominated, and no one could quite decide which was better than the other. In more recent years, alternatives have emerged, and we are now left with a plethora of wonderful methods to choose from.

Examples

Example 1: 735 – 491

Example 2: 4203 – 2984

Vocabulary Check

minuend – subtrahend = difference

The word 'subtract' comes from the Latin *sub-* ('from below') and *trahere* ('to draw').

Before 1800, the words 'subduction' and 'substraction' were sometimes used instead of 'subtraction'.

19th-century vocabulary **Box 1.1**

The language we use to describe subtraction hasn't changed much over the centuries. This extract is from the American textbook *Higher Arithmetic* (Wentworth and Smith, 1919).

> **Language of Subtraction.** The Latin for "number to be diminished" is *numerous minuendus*, and that for "number to be taken from under" is *numerus subtrahendus*. From these two expressions came our words "minuend" and "subtrahend." The result in subtraction has been called by various names. Of these we still use the words "remainder" and "difference." Formerly the word "rest" was used for the result, and we often to-day use such an expression as "keep the rest." In bookkeeping the word "balance" is commonly used, as in the expression "How much is my balance?"
>
> The symbol - is read "minus," the word being the Latin for less.

Method A: Decomposition Method

Example 1: 753 – 491

Write out the subtrahend under the minuend, lining up the digits so that the hundreds, tens and ones digits of each number are in the same columns respectively.

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline \end{array}$$

Start at the right. Subtract 1 from 3.

Write the difference at the bottom, in the ones column.

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline & & & 2 \end{array}$$

Now we want to 'subtract 9 from 5'. Note that because we're in the tens column now, we're actually subtracting 90 from 50.

To do this without using negatives we exchange the 700 (i.e. we see 700 as 600 + 100). We do this by crossing out the 7 in the hundreds column and replacing it with a 6. Then we write a small 1 to the left of the 5 in the tens column, giving us 15 tens. What we've done is add one of our hundreds to the 50, meaning we can now to subtract 90 from 150.

This used to be commonly referred to as 'borrowing 1 from the 7' but teachers increasingly avoid the word 'borrow', partly because 'to borrow' suggests that it will later be replaced. The word borrow has been used in this context for centuries, but the words exchange, decompose or deconstruct are now preferred.

$$\begin{array}{cccc} & \not{7}\,6 & {}^{1}5 & 3 \\ - & 4 & 9 & 1 \\ \hline & & & 2 \end{array}$$

Now we subtract 9 from 15 which gives a difference of 6 for the tens column (i.e. 150 – 90= 60).

$$\begin{array}{cccc} & \not{7}\,6 & {}^{1}5 & 3 \\ - & 4 & 9 & 1 \\ \hline & & 6 & 2 \end{array}$$

Finally we subtract the digits in the hundreds column. We've used one of our hundreds already so we have 6 left. We do 6 – 4 and write a 2 at the bottom. (This actually means 600 – 400 = 200.)

$$\begin{array}{cccc} & \not{7}\,6 & {}^{1}5 & 3 \\ & 4 & 9 & 1 \\ \hline & 2 & 6 & 2 \end{array}$$

$\therefore 753 - 491 = 262$

What we effectively did here was rewrite 753 as 600 + 150 + 3.

Example 2: 4203 – 2984

Write out the subtrahend under the minuend, lining up the digits so that the thousands, hundreds, tens and ones digits of each number are in the same columns respectively.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline \end{array}$$

Start at the right. We want to subtract 4 from 3 so we will have to exchange the tens in order to do this. Unfortunately there are no tens, so instead we'll exchange the hundreds.

We do this by crossing out the 2 in the hundreds column and replacing it with a 1. Then we write a small 1 to the left of the 0 in the tens column. We have now have one hundred in our tens column (i.e. ten tens).

$$\begin{array}{ccccc} & 4 & \cancel{2}\,1 & {}^{1}0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline \end{array}$$

We are still trying to subtract our ones but now we have something in the tens column that we can exchange. Exchange the tens by crossing out the 10 and replacing it with a 9. Then we write a small 1 to the left of the 3 in the ones column, in order to add 10 to the 3.

$$\begin{array}{ccccc} & 4 & \cancel{2}\,1 & \cancel{{}^{1}0}\,9 & {}^{1}3 \\ - & 2 & 9 & 8 & 4 \\ \hline \end{array}$$

Now we are ready to subtract. We subtract 4 from 13 which gives a difference of 9 for the ones column.

$$\begin{array}{ccccc} & 4 & \cancel{2}\,1 & \cancel{{}^{1}0}\,9 & {}^{1}3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & & & 9 \end{array}$$

Then we subtract 8 from 9 which gives a difference of 1 for the tens column (we actually did 90 − 80 = 10).

$$\begin{array}{ccccc} & 4 & \cancel{2}\,1 & \cancel{{}^{1}0}\,9 & {}^{1}3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & & 1 & 9 \end{array}$$

Now we're stuck again because we can't subtract 9 from 1 in the hundreds column. So we exchange the thousands. Cross out the 4 and replace it with a 3. Then we write a small 1 to the left of the 1 in the hundreds column.

$$\begin{array}{ccccc} & \cancel{4}\,3 & \cancel{2}\,{}^{1}1 & \cancel{{}^{1}0}\,9 & {}^{1}3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & & 1 & 9 \end{array}$$

Then we subtract 9 from 11 which gives a difference of 2 for the hundreds column (i.e. 1100 – 900 = 200). So we write a 2 at the bottom. And we subtract 2 from 3 which gives a difference of 1 for the thousands column (i.e. 3000 – 2000 = 1000).

$$\begin{array}{ccccc} & \cancel{4}\,3 & \cancel{2}\,{}^{1}1 & \cancel{{}^{1}0}\,9 & {}^{1}3 \\ - & 2 & 9 & 8 & 4 \\ \hline & 1 & 2 & 1 & 9 \end{array}$$

$$\therefore 4203 - 2984 = 1219$$

What we did here was rewrite 4203 as 3000 + 1100 + 90 + 13.

Jo says: The Decomposition Method, also known as 'Borrow and Regroup', is currently taught to the vast majority of school children in England. It goes back to at least the 1200s. It was brought into its present form in the 15th century under the name a *danda* ('by giving').

For decades the Decomposition Method has been the most popular method for subtraction in schools in England. Until the mid-1900s, most children were also taught the Equal Addition Method (Method B), and there were numerous studies over the years attempting to determine which of the two methods was better. Clearly in England the Decomposition Method 'won' as the other is now rarely used.

The Decomposition Method is an efficient mechanical approach that tends to present difficulties only when there is a zero in the minuend. There is some question as to whether children follow the procedure without understanding what's going on.

Explanations **Box 1.2**

In *Methods of Teaching Arithmetic: A Lecture Addressed to the London Association of Schoolmistresses* (1869), Sir Joshua Girling Fitch expressed his frustrations with the way arithmetic was taught. Of the words 'borrow' and 'carry', he says,

> *Language like this, which simulates explanation and is yet utterly unintelligible, is an insult to the understanding of a child; it would be far better to tell him at once that the process is a mystery, than to employ words which profess to account for it, and which yet explain nothing.*

He goes on to make suggestions regarding how the procedure can be properly explained so that children understand what's going on. He urges teachers ('schoolmistresses') to explain to children that in order to do the subtraction 853 – 479, *You have simply resolved*

800 + 50 + 3,

for your own convenience, into the form

700 + 140 + 13,

and left the subtrahend 479 untouched

853	=	7 hundreds	+	14 tens	+	13
479	=	4 ,,	+	7 ,,	+	9
374		3		7		4

Verification **Box 1.3**

It was common for authors of textbooks from the 1800s to verify the result of each calculation immediately. In this extract from *The Common School Arithmetic* (Eaton, 1876), we can see that an addition is performed underneath each subtraction in order to check the result.

53. PROOF. *Add the subtrahend and the remainder together, and the sum should be the minuend.*

NOTE 1. This proof rests upon the self-evident truth, that *the whole of a thing is equal to the sum of all its parts;* thus, the *minuend* is separated into the *two parts, subtrahend and remainder; hence* the *sum* of those parts *must be* the *minuend.*

Ex. 18.

Minuend,	6 8 7 4 5
Subtrahend,	2 6 8 5 4
Remainder,	4 1 8 9 1
Proof,	6 8 7 4 5

As the sum of the subtrahend and remainder is the minuend, the work is supposed to be right.

	19.	**20.**	**21.**
Minuend,	9 8 7 5	5 3 2 7 6 9	5 7 8 4 2 6 8
Subtrahend,	2 6 5	2 7 8 4 9 3	3 2 9 6 4 1 6
Remainder,	9 6 1 0	2 5 4 2 7 6	
Proof,	9 8 7 5	5 3 2 7 6 9	

Method B: Equal Addition Method

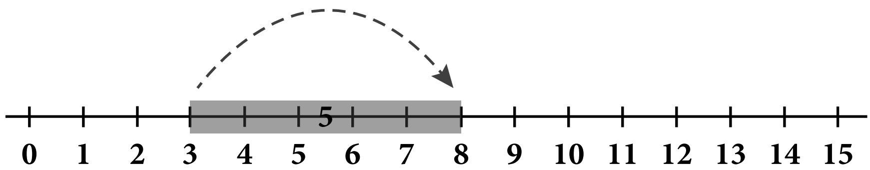
Here we need to understand the principle that if we add or subtract the same number from both the minuend and subtrahend then the difference doesn't change.

$$A - B = C$$

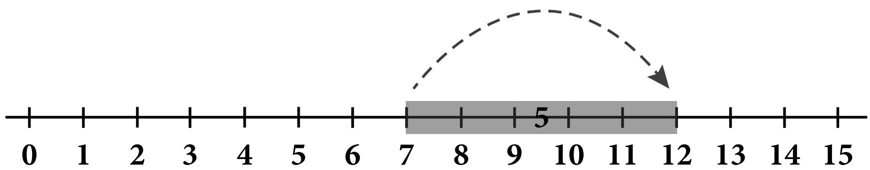
$$(A + x) - (B + x) = C$$

It's a good idea to demonstrate this on a number line.

Here we can see that the difference between 8 and 3 is 5.



If we add 4 to both numbers, we've essentially translated our subtraction up the number line; the difference remains the same.



Example 1: 753 – 491

Write out the subtrahend under the minuend, lining up the digits so that the hundreds, tens and ones digits of each number are in the same columns respectively.

	7	5	3
–	4	9	1

Start at the right. Subtract 1 from 3.

Write the difference at the bottom, in the ones column.

	7	5	3
–	4	9	1
			2

Now we want to 'subtract 9 from 5'. Note that because we're in the tens column now, we're actually subtracting 90 from 50.

To do this without using negatives we add 100 to both the minuend and subtrahend. Adding the same number to each doesn't affect the difference.

$$753 - 491 = (753 + 100) - (491 + 100)$$

The placement of the extra hundreds is as follows: put a small 1 next to the 5 in the minuend, so we have 15 tens instead of 5 tens. At the same time, cross out the 4 in the subtrahend so we have 500 instead of 400.

	7	$^{1}5$	3
−	~~4~~ 5	9	1
			2

Now we can subtract 9 from 15 and put a 6 in the tens column of the difference.

	7	$^{1}5$	3
−	~~4~~ 5	9	1
		6	2

And finally we can subtract 5 from 7 and put a 2 in the hundreds column of the difference.

	7	$^{1}5$	3
	~~4~~ 5	9	1
	2	6	2

$$\therefore 753 - 491 = 262$$

So here we added 100 to 753 and wrote it as 700 + 150 + 3. We also added 100 to 491 and wrote it as 500 + 90 + 1. By adding 100 to both minuend and subtrahend, the difference did not change.

Example 2: 4203 – 2984

Write out the subtrahend under the minuend, lining up the digits so that the hundreds, tens and ones digits of each number are in the same columns respectively.

	4	2	0	3
−	2	9	8	4

Start at the right. We want to subtract 4 from 3. To do this without using negatives we add 10 to both the minuend and subtrahend. Adding the same number to each doesn't affect the difference.

$$4203 - 2984 = (4203 + 10) - (2984 + 10)$$

The placement of the extra tens is as follows: put a small 1 next to the 3 in the minuend, so we have 13 ones instead of 3 ones. At the same time, cross out the 8 in the subtrahend so we have 90 instead of 80.

$$\begin{array}{ccccc} & 4 & 2 & 0 & {}^{1}3 \\ - & 2 & 9 & \cancel{8}\,9 & 4 \\ \hline \end{array}$$

Now we can subtract 4 from 13 and put a 9 in the ones column of the difference.

$$\begin{array}{ccccc} & 4 & 2 & 0 & {}^{1}3 \\ - & 2 & 9 & \cancel{8}\,9 & 4 \\ \hline & & & & 9 \end{array}$$

Now we want to subtract 9 tens from 0 tens. So we add 100 to both the minuend and subtrahend. To do this, put a small 1 next to the 0 in the minuend, so we have 100 instead of 0. At the same time cross out the 9 in the subtrahend so we have 1000 instead of 900.

$$\begin{array}{ccccc} & 4 & 2 & {}^{1}0 & {}^{1}3 \\ - & 2 & \cancel{9}\,10 & \cancel{8}\,9 & 4 \\ \hline & & & & 9 \end{array}$$

Now we can subtract 9 tens from 10 tens and put a 1 in the tens column of the difference.

$$\begin{array}{ccccc} & 4 & 2 & {}^{1}0 & {}^{1}3 \\ - & 2 & \cancel{9}\,10 & \cancel{8}\,9 & 4 \\ \hline & & & 1 & 9 \end{array}$$

Now we want to subtract 10 hundreds from 2 hundreds. So we add 1000 to both the minuend and subtrahend. To do this put a small 1 next to the 2 in the minuend, so we have 1200 instead of 200. At the same time cross out the 2 in the subtrahend so we have 3000 instead of 2000.

$$\begin{array}{ccccc} & 4 & {}^{1}2 & {}^{1}0 & {}^{1}3 \\ - & \cancel{2}\,3 & \cancel{9}\,10 & \cancel{8}\,9 & 4 \\ \hline & & & 1 & 9 \end{array}$$

Finally subtract 10 hundreds from 12 hundreds and write 2 in the hundreds column, and subtract 3 thousands from 4 thousands and write 1 in the thousands column.

	4	$^{1}2$	$^{1}0$	$^{1}3$
−	~~2~~ 3	~~9~~ 10	~~8~~ 9	4
	1	2	1	9

∴ 4203 − 2984 = 1219

So here we added 1110 to 4203 and wrote it as 4000 + 1200 + 100 + 13. We also added 1110 to 2984 and wrote it as 3000 + 1000 + 90 + 4. By adding 1110 to both minuend and subtrahend, the difference did not change.

Although I've explained each individual step, in reality pupils subtracting using this method would do so mechanically like they do with Method A, quickly moving between numbers following a set procedure.

Jo says: For centuries, until the mid-1900s, most children in England were taught the Equal Addition Method, which was normally known as 'Borrow and Pay Back'. It was taught either alongside or instead of the Decomposition Method (Method A). There were numerous studies over the years attempting to determine which of the two methods was better. The Equal Addition Method, which has been around since the 1400s, was considered by many to be the most efficient, particularly when the minuend contains zeros. For a long time, it was the most popular method, but in England it is now rarely taught.

Alternative approaches in the 19th century **Box 1.4**

In this extract from *The Common School Arithmetic* (Eaton, 1876), we can see that children were taught both the Decomposition Method and the Equal Addition Method.

17. From 483 take 257.

OPERATION.

Minuend,	4 8 3
Subtrahend,	2 5 7
Remainder,	2 2 6

There are two methods of explaining this operation:

1st. As we cannot take 7 units from 3 units, *one* of the 8 *tens* is put with the 3 units, making 13 units, and then, 7 units from 13 units leave 6 units. Now as *one* of the 8 *tens* has been put with the 3 *units* only 7 tens remain in the minuend, and 5 tens from 7 tens leave two tens, and, finally, 2 hundreds from 4 hundreds leave 2 hundreds; ∴ the entire remainder is 226.

2d. Instead of *taking away* 1 of the 8 tens in the minuend, we may *add* 1 ten to the 5 tens in the subtrahend, and then take the *sum* (6 tens) from the 8 tens, since the reult is 2 tens by either process.

The second mode depends on the principle, *that, if two numbers are equally increased, the difference between them remains unchanged;* thus, the difference between 9 and 4 is 5, and, if 10 is added to both 9 and 4, making 19 and 14, the difference still is 5. Now, in solving Ex.17 by the second method, we add 10 *units* to the *minuend* and 1 *ten* (*the same as* 10 *units*) to the *subtrahend*, and ∴ find the *same remainder* as by the first method.

Method C: Expanded Form Method

This method is identical to the Decomposition Method (Method A) except we show the decomposition much more clearly.

Example 1: 753 – 491

Write out the subtrahend and the minuend as a sum of their hundreds, tens and ones.

753 = 700 + 50 + 3

491 = 400 + 90 + 1

Now set them out in columns.

	H	T	O
	700	50	3
–	400	90	1

Start at the right. Subtract 1 from 3.

Write the difference at the bottom, in the ones column.

	H	T	O
	700	50	3
–	400	90	1
			2

Now we want to subtract 90 from 50. To do this without using negatives we can exchange. Instead of writing 750 as 700 + 50, we write it as 600 + 150.

	H	T	O
	6~~7~~00	150	3
–	400	90	1
			2

Now we can subtract 90 from 150.

	H	T	O
	6~~7~~00	150	3
–	400	90	1
		60	2

And subtract 400 from 600.

	H	T	O
	6~~7~~00	150	3
–	400	90	1
	200	60	2

The difference is the sum of the numbers in the bottom row.

$$\therefore 753 - 491 = 262$$

Example 2: 4203 – 2984

Write out the subtrahend and the minuend as a sum of their thousands, hundreds, tens and ones.

$$4203 = 4000 + 200 + 3$$

$$2984 = 2000 + 900 + 80 + 4$$

Now set them out in columns.

	Th	H	T	O
	4000	200	0	3
–	2000	900	80	4

Start at the right. We want to subtract 4 from 3. To do this without using negatives we can exchange the hundreds. Instead of writing 4203 as 4000 + 200 + 3, we write it as 4000 + 100 + 100 + 3.

	Th	H	T	O
	4000	1~~2~~00	100	3
–	2000	900	80	4

And then we exchange the tens so we have 4000 + 100 + 90 + 13.

	Th	H	T	O
	4000	1~~2~~00	9~~10~~0	13
–	2000	900	80	4

Now we complete the difference in the ones and tens columns by subtracting.

	Th	H	T	O
	4000	1~~2~~00	9~~10~~0	13
–	2000	900	80	4
			10	9

In the hundreds column we want to subtract 900 from 100, so we exchange the thousands. Instead of writing 4100 as 4000 + 100, we write it as 3000 + 1100.

	Th	H	T	O
	3~~4~~000	11~~2~~00	9~~10~~0	13
−	2000	900	80	4
			10	9

Now we can fill in the differences in the hundreds and thousands columns.

	Th	H	T	O
	3~~4~~000	11~~2~~00	9~~10~~0	13
−	2000	900	80	4
	1000	200	10	9

The difference is the sum of the numbers in the bottom row.

$$\therefore 4203 - 2984 = 1219$$

> Jo says: This is the same as the Decomposition Method (Method A). Although slightly less efficient (particularly for very large numbers), it makes it really clear what's going on.

Method D: Partitioning Method

Example 1: 753 − 491

Partition the subtrahend.

$$491 = 400 + 90 + 1$$

Subtract the subtrahend in parts. Start by subtracting 400 from 753.

$$753 - 400 = 353$$

Now subtract 90 from 353. If that's too difficult, then split 90 into 50 + 40. First subtract 50.

$$353 - 50 = 303$$

Then subtract 40.

$$303 - 40 = 263$$

Finally, subtract 1 from 263.

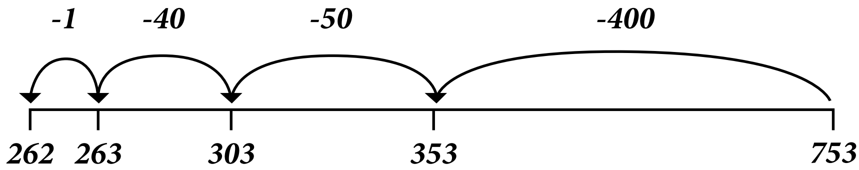
$$263 - 1 = 262$$

$$\therefore 753 - 491 = 262$$

To keep track of our workings, we could cross each part of the subtrahend off as we subtract.

The steps do not have to be done in the order described.

An empty number line (i.e. a blank number line with no markers or scale) can provide a helpful visualisation here, though is not essential.



Example 2: 4203 – 2984

Partition the subtrahend.

$$2984 = 2000 + 900 + 80 + 4$$

Subtract the subtrahend in parts. Start by subtracting 2000 from 4203.

$$4203 - 2000 = 2203$$

Now subtract 900 from 2203. If that's too difficult, then split 900 into 200 + 700. First subtract 200.

$$2203 - 200 = 2003$$

Then subtract 700.

$$2003 - 700 = 1303$$

Next subtract 80 from 1303.

$$1303 - 80 = 1223$$

Finally subtract 4 from 1223. If that's too difficult, then split 4 into 3 + 1. First subtract 3.

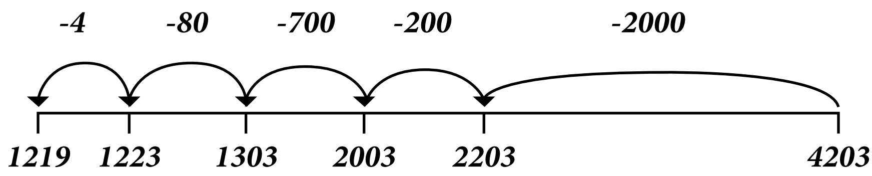
$$1223 - 3 = 1220$$

And finally subtract 1.

$$1220 - 1 = 1219$$

$$\therefore 4203 - 2984 = 1219$$

Again, an empty number line can provide a helpful visualisation here.



Jo says: In this method we have only partitioned the subtrahend. It is similar to the Expanded Form Method (Method C) where both minuend and subtrahend were partitioned. Most children will be very familiar with partitioning by the time they meet subtraction. This method, which is often introduced using a number line as a visual aid, is conceptually clear but more time consuming than column methods.

Method E: Counting Up Method

Example 1: 753 – 491

Write out the subtrahend under the minuend.

	7	5	3
–	4	9	1

Work out what we need to add to 491 to get 753. Do this in stages.

First we can add 9 to 491. This gives 500.

	7	5	3	
–	4	9	1	
			9	(500)

At each stage, write what you've added and keep a running total so you know when you've reached 753.

Next we add 200 to 500. This gives 700.

	7	5	3	
–	4	9	1	
			9	(500)
	2	0	0	(700)

Next add 50 to 700. This gives 750.

	7	5	3	
−	4	9	1	
			9	(500)
	2	0	0	(700)
		5	0	(750)

Next add 3 to 750. This gives 753 as required.

	7	5	3	
−	4	9	1	
			9	(500)
	2	0	0	(700)
		5	0	(750)
			3	(753)

Now sum the numbers you added to 491.

	7	5	3	
−	4	9	1	
			9	(500)
	2	0	0	(700)
		5	0	(750)
			3	(753)
	2	6	2	

If 491 + 262 = 753 then 753 − 491 = 262.

Example 2: 4203 – 2984

Write out the subtrahend under the minuend.

	4	2	0	3
−	2	9	8	4

Work out what we need to add to 2984 to get 4203. Do this in stages.

First add 6 to 2984. This gives 2990.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)

At each stage write what you've added and keep a running total so you know when you've reached 4203. Note that the choice of numbers and order that they are added is unimportant as long as we arrive at the minuend eventually.

Next add 10 to 2990. This gives 3000.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)
			1	0	(3000)

Next add 1000 to 3000. This gives 4000.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)
			1	0	(3000)
	1	0	0	0	(4000)

Next add 200 to 4000. This gives 4200.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)
			1	0	(3000)
	1	0	0	0	(4000)
		2	0	0	(4200)

Finally add 3 to 4200. This gives 4203 as required.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)
			1	0	(3000)
	1	0	0	0	(4000)
		2	0	0	(4200)
				3	(4203)

Now find the total of the numbers you added to 2984.

	4	2	0	3	
−	2	9	8	4	
				6	(2990)
			1	0	(3000)
	1	0	0	0	(4000)
		2	0	0	(4200)
				3	(4203)
	1	2	1	9	

If 2984 + 1219 = 4203 then 4203 − 2984 = 1219.

Jo says: This method used to be referred to as the 'Austrian Method'. It's the way a shopkeeper might mentally work out how much change to give.

It's really clear to see how this method works. It requires number sense as well as the ability to add. It is sometimes taught as an initial method for subtraction (often using a number line), but then superseded by the more procedural Decomposition Method (Method A) which is more efficient. In some parts of the world this method is becoming increasingly popular and even starting to replace the Decomposition Method as the primary method taught for subtraction.

Method F: Constant Difference Method

As in Method B, we need to understand that if we add or subtract the same number from both the minuend and subtrahend then the difference doesn't change.

$$A - B = C$$

$$(A + x) - (B + x) = C$$

Example 1: 753 – 491

Here we are trying to avoid any carrying or exchanging – the problem is the middle digit in the subtrahend being a 9. So we could add 10 to both the minuend and the subtrahend to make the subtraction easier.

$$753 - 491$$

$$= (753 + 10) - (491 + 10)$$

$$= 763 - 501$$

We write out the new subtraction in the usual column format. This isn't totally necessary though because we just need to subtract the hundreds from the hundreds, the tens from the tens and the ones from the ones. We can work left to right or right to left (because no exchanges are required), with or without the column format.

	7	6	3
	5	0	1
	2	6	2

$$\therefore 753 - 491 = 262$$

The alternative here would have been to add 9 to minuend and subtrahend and therefore subtract 500 from 762. This is probably similar to how most people would do this calculation if asked to do it in their head: subtract 500 then add 9.

Example 2: 4203 – 2984

Consider what we can usefully add to make this subtraction easier. Here, if we add 6 to both minuend and subtrahend, then we only solve the problem in the ones column.

$$4203 - 2984$$

$$= (4203 + 6) - (2984 + 6)$$

$$= 4309 - 2990$$

Instead, add 16 to both minuend and subtrahend. Now we have a far easier subtraction.

$$4203 - 2984$$

$$= (4203 + 16) - (2984 + 16)$$

$$= 4219 - 3000$$

It's not necessary to use a column format to do this simple subtraction

$$4219 - 3000 = 1219$$

$$\therefore 4203 - 2984 = 1219$$

Jo says: This neat method, also known as 'Same Change', provides a good way to avoid the exchanging that is often required in the Decomposition Method. It works particularly well when the minuend contains zeros. For example 4000 – 1478 is cumbersome to do using the Decomposition Method, but subtracting 1 from minuend and subtrahend gives 3999 – 1477 which is far easier to calculate. In other cases, we make **any** adjustment which makes the subtraction easier – often this adjustment makes the subtrahend a multiple of 10, 100 or 1000. This method is really similar to the Equal Additions Method (Method B) although it is less formulaic.

Pupils need good number sense and a strong understanding of the concept of subtraction and difference in order to benefit from this non-algorithmic approach.

Constant difference for negative arithmetic **Box 1.5**

The idea of constant difference can be used to teach negative arithmetic.

Whilst adding a positive number to a negative number is a relatively straightforward concept to explain with a number line, subtracting a negative number can be a rather complex idea. Imagine we want a pupil to subtract –5 from 7.

Standard practice is to write $(7) - (-5) = 7 + 5$ by conjuring a memorised rule. But there is a neat alternative which helps us understand where the rule comes from.

If we understand the idea of constant difference then we know that adding five to both numbers does not change the difference between them. So here, let's add five to each number.

$$(7) - (-5) = (7 + 5) - (-5 + 5)$$

This gives 12 – 0, which is clearly equal to 12. So here we have successfully subtracted a negative by adding the subtrahend.

In general we have:

$$(x) - (-y)$$
$$= (x + y) - (-y + y)$$
$$= x + y$$

Method G: Partial Differences Method

Example 1: 753 – 491

Write out the subtrahend under the minuend, lining up the digits so that the hundreds, tens and ones digits of each number are in the same columns respectively.

	7	5	3
–	4	9	1

We could work from either right to left or left to right here.

Starting from the right, we subtract 1 from 3 and write the difference in the ones column. 3 − 1 = 2.

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline & & & 2 \end{array}$$

Subtract 9 from 5 and write the difference in the tens column. 5 − 9 = −4.

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline & & -4 & 2 \end{array}$$

Unlike in other methods, it's fine to write a negative number in the bottom row.

Subtract 4 from 7 and write the difference in the hundreds column. 7 − 4 = 3.

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline & 3 & -4 & 2 \end{array}$$

Now calculate the value of the difference. Reading off the bottom row, we have 300 − 40 + 2.

$$300 - 40 + 2 = 262$$

$$\therefore 753 - 491 = 262$$

An alternative format for this approach is shown here:

$$\begin{array}{cccc} & 7 & 5 & 3 \\ - & 4 & 9 & 1 \\ \hline + & 3 & 0 & 0 \\ - & & 4 & 0 \\ + & & & 2 \\ \hline & 2 & 6 & 2 \end{array}$$

Here, the hundreds were subtracted, then the tens and then the ones. The partial differences are written on separate rows. Then we sum the partial differences: 300 − 40 + 2 = 262.

Example 2: 4203 – 2984

Write out the subtrahend under the minuend, lining up the digits so that the thousands, hundreds, tens and ones digits of each number are in the same columns respectively.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline \end{array}$$

Subtract 4 from 3 and write the difference in the ones column. 3 – 4 = –1. Unlike in other methods, we can write a negative number in the bottom row.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & & & -1 \end{array}$$

Subtract 8 from 0 and write the difference in the tens column. 0 – 8 = –8.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & & -8 & -1 \end{array}$$

Subtract 9 from 2 and write the difference in the hundreds column. 2 – 9 = –7.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline & & -7 & -8 & -1 \end{array}$$

Subtract 2 from 4 and write the difference in the thousands column. 4 – 2 = 2.

$$\begin{array}{ccccc} & 4 & 2 & 0 & 3 \\ - & 2 & 9 & 8 & 4 \\ \hline & 2 & -7 & -8 & -1 \end{array}$$

Now calculate the value of the difference. Reading off the bottom row, we have 2000 – 700 – 80 – 1.

This gives us another subtraction to perform, but we can do it in stages if required:

$$2000 - 700 = 1300$$

$$1300 - 80 = 1220$$

$$1220 - 1 = 1219$$

$$\therefore 4203 - 2984 = 1219$$

Using the alternative format for this approach we have:

	4	2	0	3
−	2	9	8	4
+	2	0	0	0
−		7	0	0
−			8	0
−				1
	1	2	1	9

Jo says: This logical approach avoids the complication we face in other methods when a digit in the subtrahend is greater than a digit in the minuend. Here we also avoid the falsehood that can be implied in other methods (that we can't take a bigger number from a smaller number).

This approach mirrors what we do later in algebra.

If we make $10 = x$ then we can write 753 − 491 as follows:

$$(7x^2 + 5x + 3) - (4x^2 + 9x + 1) = 3x^2 - 4x + 2$$

Substituting 10 for x in $3x^2 - 4x + 2$, we have $300 - 40 + 2 = 262$.

This method is not widely taught in primary schools in England, partly because children are taught subtraction before they are taught how to work with negative numbers.

This method is used in Vedic Maths (see Box 1.6).

What is Vedic Maths? **Box 1.6**

The book Vedic Mathematics was written by the Indian monk Swami Bharati Krishna Tirtha and first published in 1965. The book contains sixteen sutras and fifteen sub-sutras, each of which lists a mental calculation technique. The author claimed that these techniques were originally contained in the Vedas (religious texts originating in ancient India), though that claim has been refuted. Similar systems include the Trachtenberg system or the techniques mentioned in Lester Meyers's 1947 book High-speed Mathematics.

I have included a few - but not all - Vedic methods in this book. The methods are quick to learn and strikingly efficient, though not everyone's cup of tea.

Vinculum Numbers **Box 1.7**

Vinculum Numbers, also known as bar numbers, are used in Vedic Maths and elsewhere. They are numbers which have at least 1 digit which is negative. Negative digits are written with a bar (or vinculum) over or under them.

For example, we could write 8 as $1\underline{2}$, which means 10 – 2.

To convert to normal numbers, we just subtract the digits with the bar over or under them:

$$15\bar{6}3 = 1000 + 500 - 60 + 3 = 1443$$

$$23\overline{45} = 2000 + 300 - 40 - 5 = 2255$$

With practice one can convert between the two forms very quickly.

In Example 2 we could have used Vinculum Numbers as follows:

	4	2	0	3
–	2	9	8	4
	2	$\bar{7}$	$\bar{8}$	$\bar{1}$

So the difference is 2000 – 700 – 80 – 1 = 1219.

Method H: Complementary Method

Find the complement of the subtrahend and add it to the minuend. The complement is the number we add to the subtrahend to get it to the nearest power of 10.

Example 1: 753 – 491

Here we find the complement of 491 (i.e. what we add to 491 to make 1000). We can instantly write down 509 using the method described in Box 1.8.

Now we add the complement of the subtrahend to the minuend.

	7	5	3
+	5	0	9
1	2	6	2

And finally we subtract 1000 from the difference.

$$1262 - 1000 = 262$$

$$\therefore\ 753 - 491 = 262$$

Here we have worked out 753 + (1000 – 491) – 1000.

Simplifying, we get 753 + (1000 – 491) – 1000 = 753 – 491 as required.

In general, $a - b$ becomes $a + (10^n - b) - 10^n$ where n is the number of digits in b.

Example 2: 4203 – 2984

Here we find the complement of 2984 (i.e. what we add to 2984 to make 10000). We can instantly write down 7016 using the method described in Box 1.8.

Now we add the complement of the subtrahend to the minuend.

	4	2	0	3
+	7	0	1	6
1	1	2	1	9

And finally we subtract 10000 from the difference.

$$11219 - 10000 = 1219$$

$$\therefore 4203 - 2984 = 1219$$

Jo says: This method has been known since medieval times and is still used in some parts of the world. With a bit of practice we can find complements easily, and once we can do that then this method works very well. It's certainly very efficient. Addition is considered to be easier than subtraction.

Calculating complements **Box 1.8**

It's really quick to find a complement when you know how.

Starting with the ones and working right to left, you find the first non-zero digit and write down its number bond to ten. For the rest of the digits, you write down their number bonds to nine. In Vedic Maths this is the Nikhilam Navatashcaramam Dashatah Sutra: '**All from 9, last from 10**'.

For example, to find the complement of 837 we have 10 – 7 for the ones column, then we have 9 – 3 for the tens column and 9 – 8 for the hundreds column. So we have 163. This means that 837 + 163 = 1000.

Try subtracting 837 from 1000 and you'll see why this 'trick' works.

This is a helpful technique when teaching statistics and tends to impress A level pupils who don't know how you've done it so quickly! If P*(A)* = 0.4573 then I can instantly write down P*(not A)* = 0.5427 using 'all from 9, last from 10'.

One hundred years ago **Box 1.9**

This extract is from the American textbook *Higher Arithmetic* (Wentworth and Smith, 1919). Here they demonstrate the calculation 74208 − 63327 + 14292 − 18306 using the Complementary Method.

It will naturally be thought that no one would ever subtract in this way. As stated above, however, the method is used in machine calculation. It is also convenient in a case like the one here shown.

The complement of 63,327 is easily found by subtracting 7 from 10 and each of the other digits from 9, and similarly for 18,306. The complement can be written easily from left to right. Since there are two complements to 100,000 in this example, we must subtract 200,000 from the sun.

	74,208	74,208
−	63,327	36,673
+	14,292	14,292
−	18,306	81,694
	6,867	6,867

Method I: Nines Complement Method

Example 1: 753 – 491

Instead of subtracting from 753, we start by subtracting 491 from 999. It is easy to subtract any number from 999.

	9	9	9
−	4	9	1
	5	0	8

508 is the '9s complement' of 491. Now add 508 to 753.

	5	0	8
+	7	5	3
1	2	6	1

Subtract 1000 from the total, then add 1. This is equivalent to subtracting 999.

$$1261 - 1000 = 261$$

$$261 + 1 = 262$$

$$\therefore 753 - 491 = 262$$

A trick? **Box 1.10**

This seems like a trick but it's just clever use of numbers.

What we did here was:

$$999 - 491 + 753 - 1000 + 1$$
$$= 999 - 491 + 753 - 999$$
$$= -491 + 753$$
$$= 753 - 491$$

We can show this works for any subtraction $a - b$. This method can be modified for subtracting numbers of any length.

Example 2: 4203 – 2984

Start by subtracting 2984 from 9999.

	9	9	9	9
−	2	9	8	4
	7	0	1	5

Now add 7015 to 4203.

	7	0	1	5
+	4	2	0	3
1	1	2	1	8

Subtract 10000 from the total, then add 1.

$$11218 - 10000 = 1218$$
$$1218 + 1 = 1219$$
$$\therefore 4203 - 2984 = 1219$$

Jo says: Here we only have to do an easy subtraction – no exchanges or decomposition required – and an addition. Perhaps it's a bit too much of a 'trick' to be taught in schools, but the explanation of why it works is easy to understand.

Your Turn > Try each method with these examples:

a) 457 – 81

b) 642 – 379

c) 9000 – 6537

A	Decomposition Method	
B	Equal Addition Method	
C	Expanded Form Method	
D	Partitioning Method	
E	Counting Up Method	
F	Constant Difference Method	
G	Partial Differences Method	
H	Complementary Method	
I	Nines Complement Method	

Answers

a) 376

b) 263

c) 2463

Questioning from the 19th century **Box 1.11**

In *A Common School Arithmetic* (Hagar, 1872) the author presents a series of questions which are intended to check whether pupils understand the vocabulary, rules and principles of subtraction.

61. –1. What is SUBTRACTION? The difference? The minuend? The subtrahend? When the subtrahend and minuend are equal, what is their difference? What is the sign for subtraction? Illustrate its use.

2. What are the PRINCIPLES of subtraction? Why cannot 5 apples be subtracted by 8 dollars? If the minuend and subtrahend express dollars, what will the difference express? If the minuend is 8, what must be the sum of the subtrahend and difference?

3. What is the RULE for subtraction? Why is the subtrahend written under the minuend so that figures shall stand in the same column? Why begin at the right to subtract? What is the method of proof? When the subtrahend and difference are given, how may the minuend be found?

Contextual subtraction **Box 1.12**

This extract shows a small selection of the many 'practical problems' in *The Normal Elementary Arithmetic* (Brooks, 1865).

11. Minnie had 372 cents in her money-bank, and took out 164 cents to give a little beggar-girl; how many cents remained?

12. Andrew's kite arose 494 feet, and this was 58 feet higher than Peter's kite went; how high did Peter's kite fly?

13. Charlie wrote 724 words in two weeks; he wrote 346 words the first week; how many words did he write the second week?

14. Mary's new reader contains 76 pictures, and Fanny's contains 92 pictures; how many does Fanny's contain more than Mary's?

15. Two little girls picked 74 quarts of blackberries one summer; if one picked 37 quarts, how many quarts did the other pick?

16. Thomas said he counted 283 crows in his father's cornfield; he threw a stone and scared 126 away; how many then remained?

17. Floy and Eugie together took 3000 steps one day; if Floy took 1786 steps, how many steps did Eugie take?

Questions for the slate **Box 1.13**

This exercise is taken from the American textbook *Introduction to the National Arithmetic* (Greenleaf, 1845). Note the antiquated measures. Answers are presumably provided for the first seven questions to aid the teacher – the questions would have been read out loud or written on the blackboard. Slates were used extensively in Victorian schools – they were equivalent to modern day mini-whiteboards. Children wrote on their wooden-framed slates with the aid of a slate pencil. Slates were used because paper was expensive.

QUESTIONS FOR THE SLATE.

	2.	**3.**	**4.**	**5.**
	£.	Cwt.	Miles.	Bushels.
Minuend,	789	376	531	4789050
Subtrahend,	346	187	389	1789582
	443	189	142	2999468

	6.	**7.**	**8.**	**9.**
	Tons.	Gallons.	Pecks.	Feet.
From	978	67158	14711	100000
Take	199	14339	9197	90909
	779	52819		

	10.	**11.**	**12.**	**13.**
	Miles.	Dollars.	Minutes.	Seconds.
From	67895	456798	765321	555555
Take	19999	190899	177777	177777

	14.	**15.**
	Rods.	Acres.
From	100200300400500	1000000000000
Take	90807060504030	999999999999