GARAGE PHYSICS

by eisco

Spiral of Theodorus Block Kit
GP00006

Guide for Educators

The Spiral of Theodorus Block Kit allows students of almost any age to experiment with square roots and irrational numbers in physical form. This kit, though simple in design, provides a setting for age-appropriate discussions of the structure and behavior of a whole new type of number. Young children can use the kit to explore right triangles and ramps. Elementary school children can use the triangles to make their first foray into indirectly defined quantities. They can work to make Theodorus' spiral and other related shapes, and depending on their age level, elementary school children can explore the pythagorean theorem and square roots. Middle school students can use the triangles to test equivalencies with square roots, and explore abstract concepts like a sequence of geometric shapes based on the Fibonacci numbers. High school students can use the kit to explore a proof of irrationality for non-square integer roots, a method for approximating square roots and, using the Spiral of Theodorus, approximating the number π.

Contents of Kit

16 triangle shaped blocks of increasing height - Set of stickers for labeling the blocks

Ball-point pen

Background

Theodorus was a contemporary of Plato and may have been the first to prove the irrationality of the square roots from 3 to 17 (which are not perfect squares, like 4, 9, and 16). The spiral which is attributed to Theodorus is thought to have been used in his proofs. The Spiral of Theodorus allows students to explore the square roots tangibly and kinesthetically. It provides a setting for students to gain exposure to irrational numbers and indirectly defined quantities. It also has a number of fascinating properties allowing older students to make connections to the Pythagorean theorem, and an approximation of pi.

This guide provides ideas for age-appropriate discussion and interaction between educator and student.

Young Children

Free exploration

Allow the child to simply play with the blocks, exploring different ways to arrange them.

Order the blocks

You can lead the child in this direction by giving the child two blocks and asking, “Which is taller?”

Count the blocks

Count the blocks from smallest to largest saying the numbers out loud as you point. This helps promote the idea that when you count, the number you speak is associated with a single item. At this level it is unwise to insist the child do it right - rather, praise their efforts and always demonstrate the correct way to do it. They’ll catch on when they’re ready.

Elementary School

Fun designs

Use the blocks to form interesting geometric designs. Ask your child to speculate on why these shapes form such interesting patterns.

Spiral of Theodorus

Pointy Spiral of Theodorus

Ram's Horn

Sail

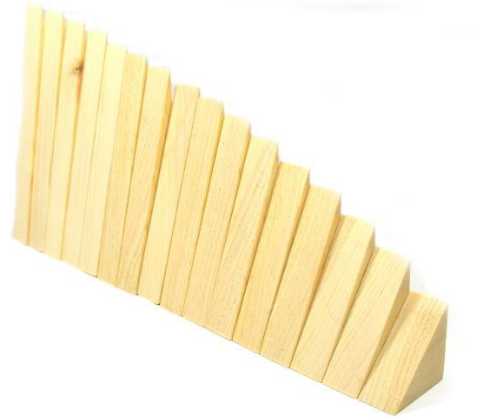
Definition of a Right Triangle

A right triangle is easy to identify in block form. If you put the short side of the triangle on a table, the medium side always points straight up.

Right triangles always have one side which is longer than the other two. This side is called the hypotenuse. A right triangle can have three sides that are all different. Can you find one which has two sides the same?

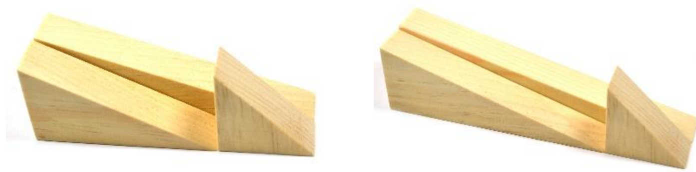
Ordering by height: Comparing heights

Put the blocks on their short side with the sharp corner pointing up. Arrange them so that the side that points straight up are facing you. Now order the blocks by height. Use the stickers to label the blocks 1 through 16, from smallest to tallest.



Now we can compare blocks. Is the 1-block and the 2-block as tall as the 3-block?

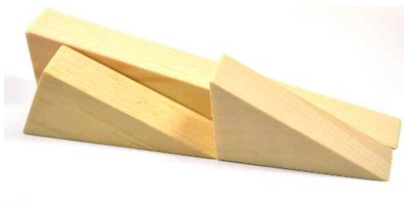
Are the 1-block and the 4-block as tall as the 9-block? What about the 1-block, the 9-block, and the 16-block?



Are there any other groups of blocks which add up to another block?

Can you also use the short sides of the triangles? Try using the 1- and 2-blocks to match the long side of the 4-block.

Now try the case where you can use any side of the triangles. Can you find combinations of blocks that equal other blocks? Try using the hypotenuse of the 3-block and the long side of the 4-block to match the long side of the 16-block.



Ordering by angle: What is an angle?

Put the blocks on their long side with sharpest corner pointing along the table. When they are sitting this way they look like ramps. Some of the ramps feel very steep, while others are not steep at all. Arrange the blocks by the steepness of the ramps. Is it the same order as when you ordered by height?

Now look at the sharpest corner of each block. Some are very sharp and some are not. Arrange the blocks by the sharpness of the corner. How does this compare the previous orderings?

When two sides meet, the space between them where they meet is called an angle. This is another way to talk about a corner on a triangle. A triangle has three angles, or three corners. Does a steep ramp have a smaller or larger angle than a long ramp? Does a tall block have a smaller or larger angle than a short block?

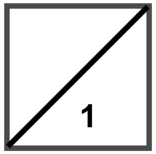
Label the blocks

Once the blocks have been ordered by height, use the stickers to label the blocks, 1-16, from smallest to largest.

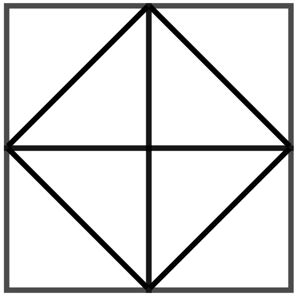
Square roots

This exercise is designed for older elementary school kids.

Take the 1-block and trace it on a piece of paper. Then flip it around and trace it again so you have a square. Label the sides of the square 1. The area of the square is the product of the sides, or 1x1=1.

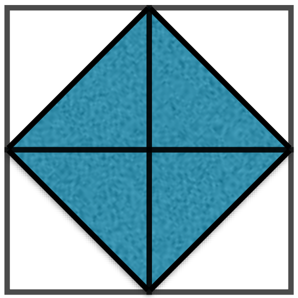


Now trace some more squares next to it using the same triangle.



What is the area of this square? (The area of the larger square is $2 \times 2 = 4$.) Notice that it is made up of eight identical triangles.

Now focus on the center square.



How many triangles are in this square? What is the area of this square? How does it compare to the area of the 8-triangle square?

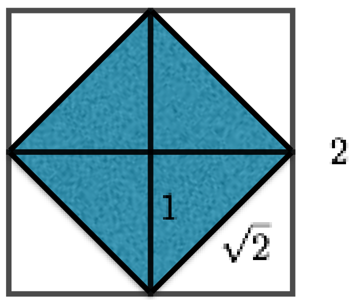
Now, we don't know the length of the sides of this square yet. But we know it's a square so the sides all have the same length. And we know that the product of two sides is equal to 2, because that is the area. So we'll invent a special symbol to express this.

$$\sqrt{2}$$

This symbol is a secret code which means, "This number has the property that when it is multiplied by itself it equals 2." We can write it this way,

$$\sqrt{2} \times \sqrt{2} = 2.$$

Now we can label the picture with our special symbol.

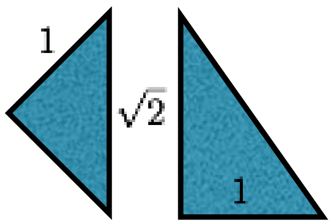


We call our special symbol ‘root 2’. We don’t know the value of root 2, but we know what it does, and that’s enough for now.

Pythagorean Theorem

This exercise is designed for older elementary school kids.

All of the triangles in the kit are right trangles. Find the 1-block and the 2-block. How is the long side of the 2-block related to the sides of the 1-block?



The Pythagorean theorem says that if we square the sides of a right triangle and then add them, they are equal to the square of the hypotenuse. What is the square of the hypotenuse of the 2-block? (3, since 1 + 2 = 3.)

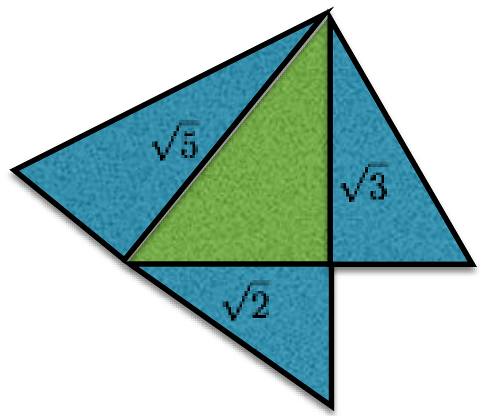
We can invent a symbol just like before to represent the number whose square is equal to 3.

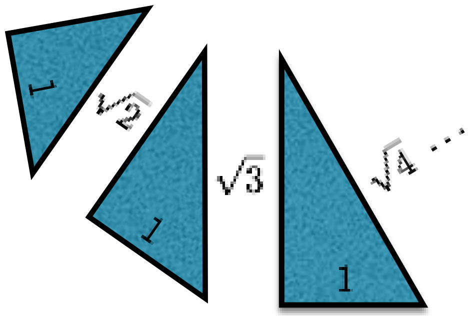
$$\sqrt{3}$$

In other words,

$$\sqrt{3} \times \sqrt{3} = 3.$$

Our symbol is called root 3. The same process can be used to work out the sides of the other blocks in order.





Each new symbol we invent has the same property, for example,

$$\sqrt{4} \times \sqrt{4} = 4,$$
$$\sqrt{5} \times \sqrt{5} = 5,$$

and so on.

It's easy to see the pattern. Use the stickers to label all the sides of the triangles with the appropriate special symbols.

Pythagorean algebra

We can use the sides of the triangles as rulers to draw other triangles. Can you make a triangle with the long sides of the 2-block, the 3-block, and the 5-block?

What is special about this triangle? Does it follow the Pythagorean theorem?

It's easy to check that this works for any combination of blocks whose labels add. For example the long sides of the 3-block, the 5-block, and the 8-block make a right triangle. Try to make more right triangles in this way.

Middle School

In addition to the material above, middle school students can explore sequences and relationships between multiples of irrational numbers.

Fibonacci Triangles

A sequence is an ordered list of numbers. The Fibonacci sequence is a famous sequence which starts with the numbers 0 and 1, and has the property that every element of the list is the sum of the previous two elements. For example,

{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …}

The Fibbonacci sequence has the remarkable property that the ratio between neighboring elements approaches the Golden ratio,

$$\varphi = \frac{1+\sqrt{5}}{2}$$

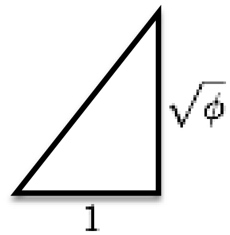
We can imagine a different sequence based on the Fibonacci sequence,

{0, √1, √1, √2, √3, √5, …}

Any three neighboring elements form the sides of a right triangle which we'll call a Fibonacci triangle. Can you explain why this is true?

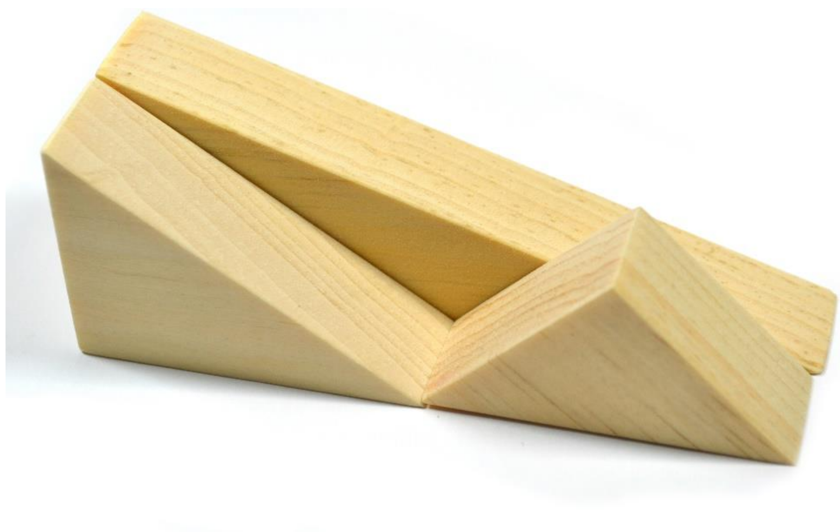
How do the Fibonacci triangles change as you use they get larger and larger? How do their angles change? Use the blocks as rulers to trace out larger and larger Fibonacci triangles.

Draw the limiting Fibonacci triangle with the shortest side rescaled to equal 1.



<u>Multiples of square roots</u>

Put the hypotenuse of the 1-block and the long side of the 2-block end to end. Is this equal to any other side of a single block? With which blocks and sides can you match?



We can write

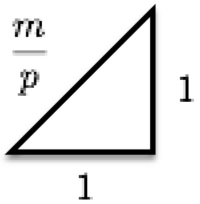
$$2\sqrt{2} = 8$$

Can you use algebra show why this is true? Can you find any other examples that work like this? There are two more examples. How would you write these relationships? Can you show that the way you've written them is correct?

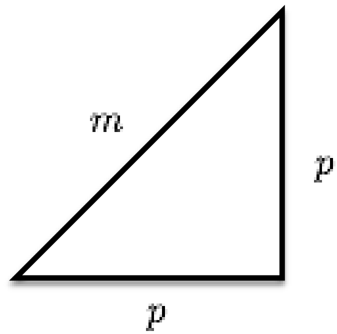
<u>HighSchool</u>

<u>Proof that √2 is irrational</u>

A number is rational if it can be expressed as a fraction of integers. Let's assume this is possible for root 2. We can then label the drawing



where the fraction is in reduced form. This means that the integers m and p do not have any common prime factors. We can draw a helpful similar triangle.



Then we can use the Pythagorean theorem to write a relationship between m and p,

$$m^2 = 2p^2$$

But this means that m and p have some common prime factors (except if p=1, but p=1 is not an option because then m is not an integer). This is a contradiction because we already said they did not have any common prime factors. Consequently, root 2 cannot be written as a fraction of integers, and is therefore irrational.

Can you use the same ideas to prove that root 3 is irrational? Does the same proof work for root 4? Why not? What about other integer roots?

Estimating the square roots

One clever way to estimate the value of an integer square root is to make a guess and use the following formula,

$$g_n + 1 = \frac{g_n + N}{g_n + 1}$$

to find a better estimate, and so on, where N is the integer whose square root we seek to estimate.

Here is an example of how this works for root 2. We purposely pick a very bad initial estimate.

$$g_0 = 3$$
$$g_1 = \frac{3+2}{3+1} = \frac{5}{4}$$

$$g_2 = \frac{\frac{5}{4} + \frac{8}{4}}{\frac{5}{4} + \frac{4}{4}} = \frac{13}{9}$$

$$g_3 = \frac{\frac{13}{9} + \frac{18}{9}}{\frac{13}{9} + \frac{9}{9}} = \frac{31}{22}$$

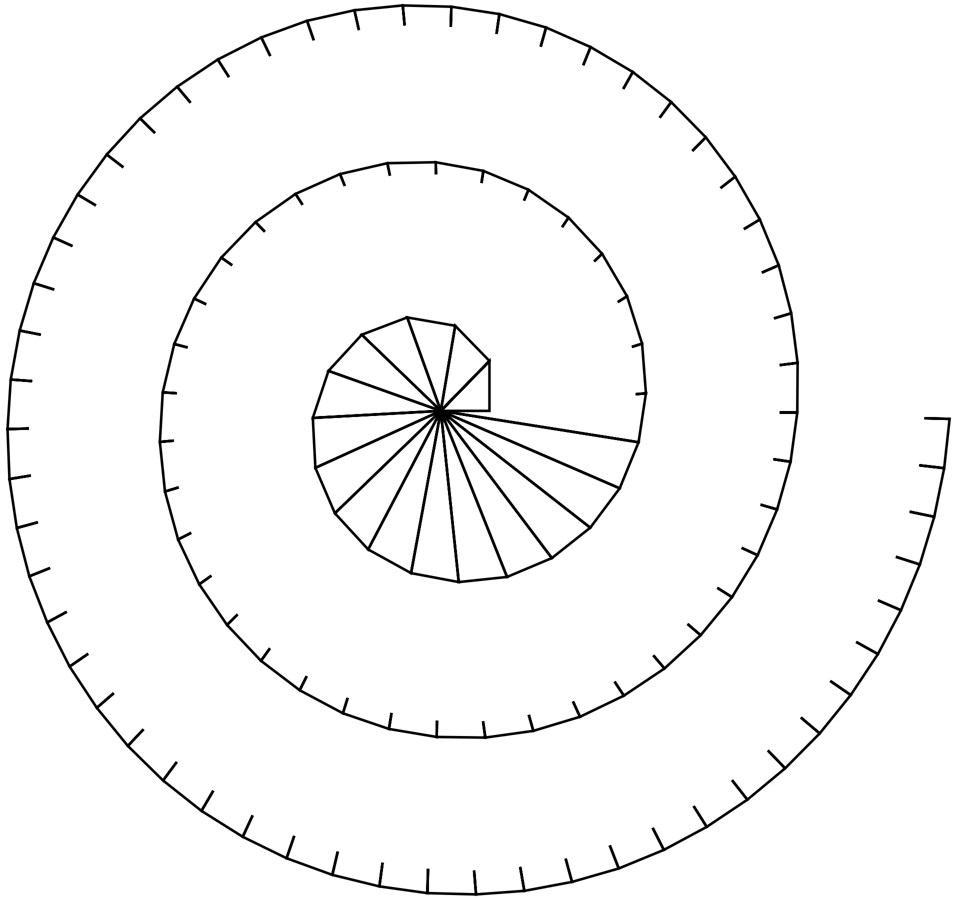
$$g_4 = \cdots$$

Use a calculator to check these estimates. Try to find good estimates of other integer square roots. Can you get the same set of estimates from two different initial guesses?

Why does this formula work? Are there any initial guesses for which the formula fails? Try this method using g_0=-1. What happens if you use root N as your initial guess?

Approximation of PI

We can continue the Spiral of Theodorus by adding new triangles. As the Spiral of Theodorus wraps around, the distance between one wrapping and the next begins to approach π. We can check this using the drawing to the right. Label the tick marks sequentially from 18. Then calculate the distance between tick marks on different windings which have nearly the same angle.



For example,

$$\sqrt{18} - 1 = 3.243\ldots$$

How close is the approximation for the last tick mark on the drawing? For reference, π = 3.14159265359.