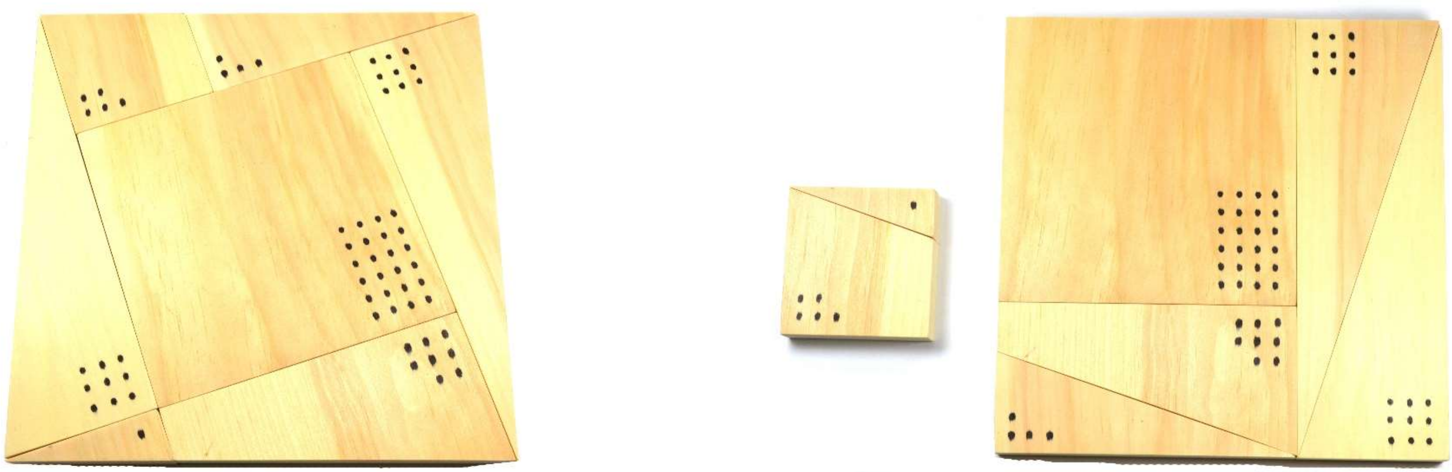
GARAGE PHYSICS

by eisco

Pythagorean Theorem Puzzle Kit
GP00004



Guide for Educators

We've designed this kit to enable an engaging back and forth for almost every level of child. Young children can use the kit to explore basic shapes and similarity. Elementary school children can engage in solving the central puzzle of the Pythagorean theorem - forming squares that represent the squares of the hypotenuse and shorter sides. Middle school students can use the puzzle to work out three algebraic proofs of the Pythagorean theorem, including the proof put forward by President James A. Garfield. High school students can use the kit to explore units of measure and reason out Einstein's proof of the Pythagorean theorem. Here we provide suggestions on how to engage with each of these age groups.

Young Children

Free exploration

Allow the child to simply play with the puzzle, exploring different ways to orient the shapes and put them together.

Naming and forming of shapes

Many shapes can be formed using the blocks in this set, including right triangle, isosceles triangle, trapezoid, parallelogram, rhombus, quadrilateral, square, rectangle, kite, and irregular pentagon. Many of these shapes will be created during free exploration. Point them out as they are created.

Meaning of similar shapes

Have a conversation about same versus similar. Some similar shapes which can be formed are triangles, parallelograms, trapezoids, quadrilaterals, and squares.

Elementary School

Meaning of right triangle

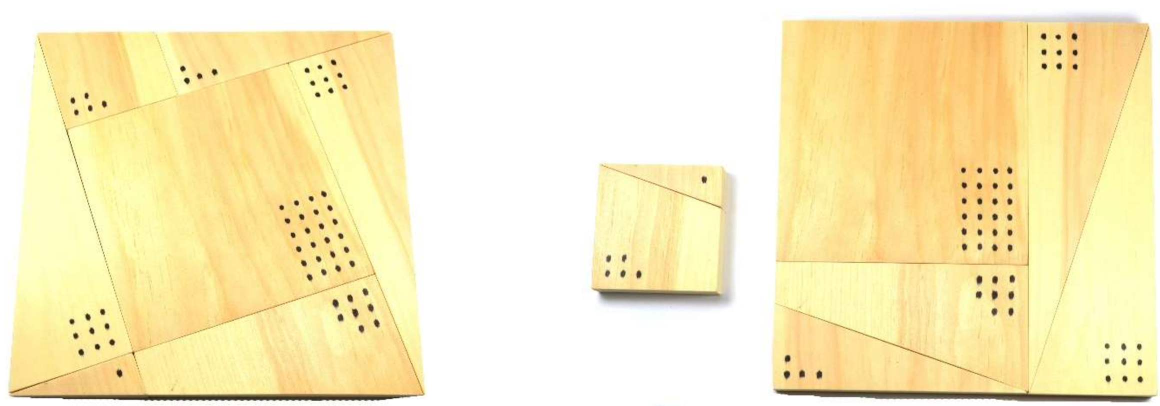
Pull out the largest triangle and explain that this is a special kind of triangle. You can explain that if you put one of the shorter sides down flat, the other side points straight up. This is a conceptual way of getting at the meaning of a 'right' triangle.

Pythagorean theorem

Explain that Pythagoras discovered a trick to relate the legs of a triangle to its hypotenuse which applies to every right triangle:

"If you form squares with sides equal to the shorter sides of a right triangle, you can break those squares up and use them to create one big square with sides equal to the longest side of the triangle."

In working with our own children, we have found it helpful to point out one of the short sides and then to draw the largest triangle and a square next to it (on paper, or using sidewalk chalk) that is that size. Then the challenge is to fill the drawn square. The difficulty obviously increases from smallest square to largest square. One way to forestall frustration if a child gets stuck is to give hints by pointing to shapes that are correctly placed. This narrows down options for finishing the puzzle.



The left image shows the c^2 square. The right image shows the a^2 and b^2 squares.

Understanding Area

Count the dots on each piece and relate them to the size of each piece. Small pieces have fewer dots, big pieces have more. Trace the 1-dot triangle on a piece of paper and then cut out five copies of it. Use the paper triangles to tile the larger triangles to visually see why they have more dots. For example, the 4-dot triangle 1can be tiled with four copies of the 1-dot triangle. The 5-dot quadrilateral can be tiled with five 1-dot triangles. If you use the 4-dot triangle and 4 one-dot paper triangles, you can tile the 8-dot quadrilateral. In this way you can very easily show that any of the larger pieces can be tiled using the smaller pieces.

It is particularly important to point out that the area of an object is a new notion of what it means to be ‘big’. The 4-dot triangle is taller than the 5-dot quadrilateral, but has smaller area because fewer 1-dot triangles fit within it without overlapping. The 9-dot triangle is taller than the 24-dot square, but has obviously smaller area. It’s fun to show this by chopping up the 9-dot triangle into the 4-dot triangle and the 5-dot quadrilateral and showing that they fit within the 24-dot square.

If your child is confident with adding numbers up to 100 (typically 2nd grade and older), you can add up the numbers in each of the two configurations (60 for the large square, and 9 and 51 for the smaller squares).

Middle School

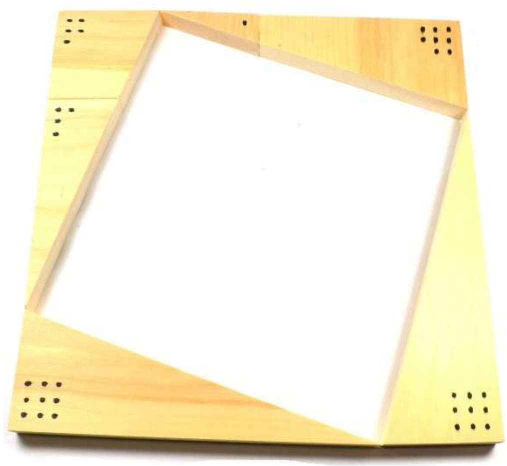
Symbolic representation

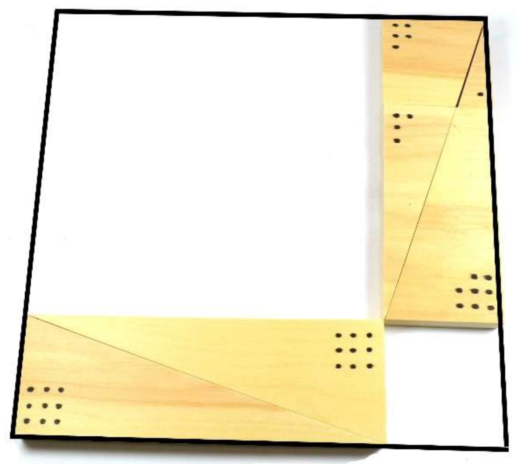
Pencil in an ‘a’, ‘b’, and ‘c’ on the main triangle and talk about how the Pythagorean theorem would look using those symbols to represent the side of a triangle ($a^2 + b^2 = c^2$). Then relate each part of the Pythagorean theorem to its corresponding square.

Algebraic proofs

Show how forming larger- and smaller-sized squares can be used to prove the theorem as shown in the figures below.

$$(a + b)^2 = a^2 + 2ab + b^2 = c^2 + 2ab, \text{ so } a^2 + b^2 = c^2$$

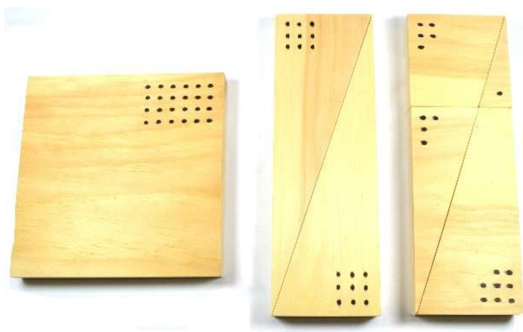


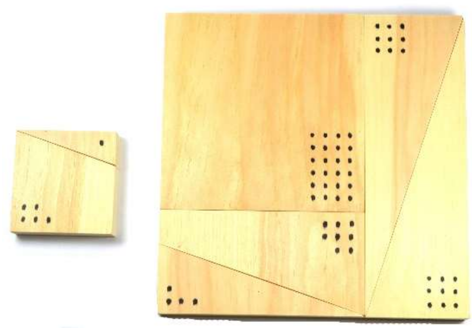


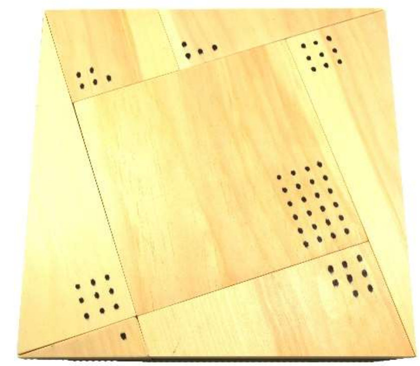
Square of sum proof

The left image shows (a + b) square with c^2 square inside and four triangles with area (a x b)/2 each. The same $(a + b)^2$ square can be rearranged with a^2 and b^2 squares with two rectangles with area (a x b).

$$(b - a)^2 + 2ab = a^2 + b^2 = c^2, \text{ so } a^2 + b^2 = c^2$$







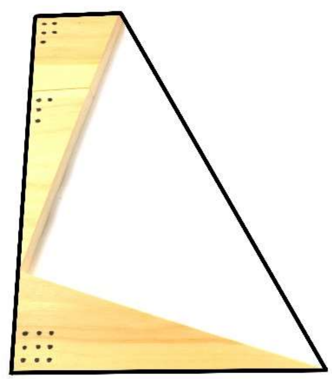
Square of difference proof

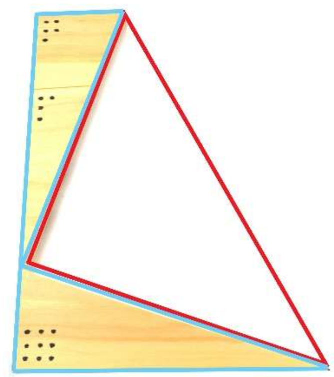
The left image shows a $(b - a)^2$ square and two ab rectangles. The pieces can be rearranged into the c^2 square (middle image), or a^2 and b^2 squares (right image).

President James A. Garfield's trapezoid proof

President James A. Garfield used the area of a trapezoid, area = height (top+base)/2, to prove the Pythagorean thorem.

$$(a+b)\ x\ \frac{(a+b)}{2} = \frac{a^2+b^2}{2} + ab = \frac{c^2}{2} + ab,\ so\ a^2 + b^2 = c^2$$





The left image shows a trapezoid with base b, top a, and height (a + b) giving area $(a + b)^2/2$. Multiplying out $(a + b)^2/2$ gives $(a^2 + b^2)/2 + a\,b$. The right image shows how the same trapezoid can be seen as a triangle with area $c^2/2$ and two triangles with area (a b)/2.

High School

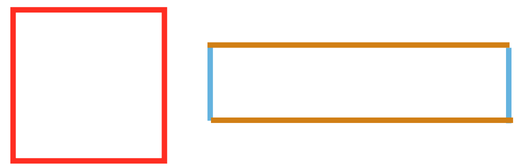
Units of length

Rather than using units of inches to determine the area, we chose non-traditional units to simplify the problem for elementary school kids. Noting that the area of the smallest square is 6 units of area, work out the lengths of sides the smallest square ($\sqrt{6}$). Then use the area of the smallest triangle (1) to work out the ratios between the sides of the triangles (3:1). Then use this ratio to work out the lengths of the other two sides of the main triangle in the non-traditional units ($b=3\sqrt{6}$, $c=\sqrt{60}$). Now get a ruler and relate the non-traditional units to inches (1 inch = $\sqrt{(2/3)}$ non-traditional units). Verify your conclusions against the total area of the b^2 and c^2 squares by counting the dots. Verify the statement of the Pythagorean theorem using the lengths you found.

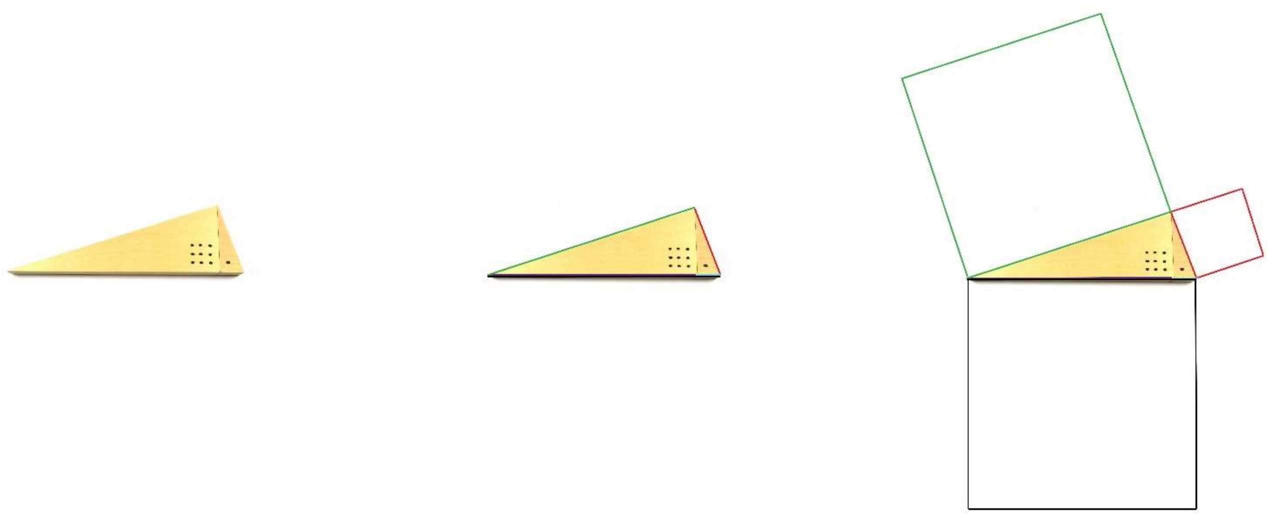
Similar triangles proof

The similar triangles proof uses three similar triangles, with the third triangle being composed of the first two triangles. This proof looks at the ratios of sides of these similar triangles.

First we point out an important geometric relationship between rectangles and squares of equal area. If you want to make a square twice as wide (brown dimension) and conserve area, you need to make it half as tall (red dimension). This works for any stretching of the square while conserving area. The ratio between the short side and the square side is the same as the ratio of the red side to the brown side. Or, more simply, blue is to red as red is to brown.



Now we look at ratios between the sides of our similar triangles using colors to code their lengths. Blue is to red as red is to brown. Yellow is to green as green is to brown.
We can use our intuition from above to draw the equal area squares and rectangles according to these statements. Since blue plus yellow equals brown, the theorem is proved.



Einstein’s proof of the Pythagorean Theorem

Einstein’s proof also uses three similar triangles, again with the third triangle being composed of the first two. If we construct a square on each triangle, we have three similar squares.

First we note that the of area each square is in direct proportion to the area of each triangle. We also know that the sum of the area of the first two triangles equals the sum of the area of the third triangle because of the way we constructed the third triangle. Therefore, it must be the case that the sum of the first two squares equals the sum of the third square. But this is just the Pythagorean theorem applied to the third triangle.