CONCEPT

6 ★ 3 relationships | TRIGONOMETRY

Main Theory & Concept

1 Reciprocal identities

ⓐ $\csc\theta = \dfrac{1}{\sin\theta}$ (←Same sign and period)

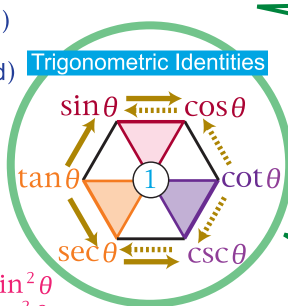
ⓑ $\sec\theta = \dfrac{1}{\cos\theta}$ (←Same sign and period)

ⓒ $\cot\theta = \dfrac{1}{\tan\theta}$ (←Same sign and period)

2 Quotient identities

ⓐ $\tan\theta = \dfrac{\sin\theta}{\cos\theta}\ \left(= \dfrac{\sec\theta}{\csc\theta}\right)$

ⓑ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}\ \left(= \dfrac{\csc\theta}{\sec\theta}\right)$



3 Pythagorean identities

ⓐ $\sin^2\theta + \cos^2\theta = ①$ ➔ $\cos^2\theta = 1 - \sin^2\theta$, $\sin^2\theta = 1 - \cos^2\theta$

ⓑ $\tan^2\theta + 1 = \sec^2\theta$ ➔ $\sec^2\theta - \tan^2\theta = ①$

ⓒ $1 + \cot^2\theta = \csc^2\theta$ ➔ $\csc^2\theta - \cot^2\theta = ①$

4 Proofs of the identities

Simplifying the trigonometric expressions

$\csc x \xrightarrow{(i)} \dfrac{1}{\sin x}$, $\sec x \rightarrow \dfrac{1}{\cos x}$, $\tan x \rightarrow \dfrac{\sin x}{\cos x}$, $\cot x \rightarrow \dfrac{\cos x}{\sin x}$ $\xrightarrow{(ii)}$

ⓐ $\sin^2x + \cos^2x = 1$ $\xrightarrow{(iii)}$ In terms of $(\sin x)$ [$\cos^2x \Rightarrow 1 - \sin^2x$]; In terms of $(\cos x)$ [$\sin^2x \Rightarrow 1 - \cos^2x$]

ⓑ $\div \cos^2x \xrightarrow{(iii)}$ In terms of $(\tan x)$

$\tan^2x + 1 \xrightarrow{(i)} \sec^2x \xrightarrow{(ii)} \dfrac{1}{\cos^2x}$, $1 + \cot^2x \rightarrow \csc^2x \rightarrow \dfrac{1}{\sin^2x}$ $\xrightarrow{(iii)}$ $\sin^2x + \cos^2x = 1$ $\xrightarrow{(iv)}$ $\sin x$, $\cos x$

Tip from Top

1 Reciprocal identities

ⓐ $\csc 10^\circ = \dfrac{1}{\sin 10^\circ}$

ⓑ $\sec 20^\circ \cdot \cos 20^\circ = 1$

ⓒ $\cot 30^\circ$ = Reciprocal of $\tan 30^\circ$

2 Quotient identities

ⓐ $\tan 40^\circ = \dfrac{\sin 40^\circ}{\cos 40^\circ}$

ⓑ $\log\cot 50^\circ = \log\dfrac{\cos 50^\circ}{\sin 50^\circ}$
$= \log\cos 50^\circ - \log\sin 50^\circ$

3 Pythagorean identities

ⓐ $\sin^2 60^\circ + \cos^2 60^\circ = 1$

ⓑ $\sec^2 70^\circ - 1 = \tan^2 70^\circ$

ⓒ $(\csc 80^\circ + \cot 80^\circ)(\csc 80^\circ - \cot 80^\circ)$
$= \csc^2 80^\circ - \cot^2 80^\circ = 1$

4 Proofs of the identities

$$\frac{\tan\theta\cdot\csc^2\theta}{1+\tan^2\theta} = \cot\theta$$

$$\text{LHS} = \frac{\tan\theta\cdot\csc^2\theta}{1+\tan^2\theta} = \frac{\tan\theta\cdot\csc^2\theta}{\sec^2\theta}$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin^2\theta}}{\dfrac{1}{\cos^2\theta}} = \frac{\left(\dfrac{1}{\sin\theta\cdot\cos\theta}\right)\cdot\cos^2\theta}{\left(\dfrac{1}{\cos^2\theta}\right)\cdot\cos^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$$

Simple Sample

1. Which of the following is equivalent to $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta$?

(A) 1 (B) $\sec\theta$ (C) $\csc\theta$
(D) $\cos\theta$ (E) $\sin^2\theta - 1$

2. If $\sin x + \cos x = 1.2$, then what is the value of $\sin x\cdot\cos x$?

(A) 0.8 (B) 0.4 (C) 0.2
(D) 0.11 (E) 0.22

3. Which of the following is equivalent to the expression of

$$\frac{1}{\sin^2\theta + \cos^2\theta + \tan^2\theta} + \frac{1}{\sin^2\theta + \cos^2\theta + \cot^2\theta}?$$

(A) 1 (B) 2 (C) 4
(D) $\sec^2\theta$ (E) $\csc^2\theta$

Essence

1. $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta}$
$= \dfrac{1}{\sin\theta} = \csc\theta$ **Ans** (C)

2. $(\sin x + \cos x)^2$
$= \sin^2x + 2\sin x\cos x + \cos^2x$
$= 1 + 2\sin x\cos x = (1.2)^2$
$\therefore 2\sin x\cos x = 1.44 - 1 = 0.44$
$\therefore \sin x\cos x = 0.22$ **Ans** (E)

3. Since $\sin^2\theta + \cos^2\theta = 1$,
$\dfrac{1}{1+\tan^2\theta} + \dfrac{1}{1+\cot^2\theta}$
$= \dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta}$
$= \cos^2\theta + \sin^2\theta = 1$ **Ans** (A)

pattern guide | Ace of Base

1 Reciprocal identities

ⓐ $\csc\theta = \frac{1}{\sin\theta}$

ⓑ $\sec\theta = \frac{1}{\cos\theta}$

ⓒ $\cot\theta = \frac{1}{\tan\theta}$

Which of the following is equivalent to the expression $\sec^2\theta + \csc^2\theta$?

(A) $\sec^2\theta \cdot \csc^2\theta$ (B) $\sin^2\theta \cdot \cos^2\theta$

(C) $\tan^2\theta + \cot^2\theta$ (D) $\sin^2\theta - \cos^2\theta$

(Sol) $\sec^2\theta + \csc^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \cdot \sin^2\theta}$

$= \frac{1}{\cos^2\theta \cdot \sin^2\theta} = \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta} = \sec^2\theta \cdot \csc^2\theta$

Ans (A)

2 Quotient identities

ⓐ $\tan x = \frac{\sin\theta}{\cos\theta}$

ⓑ $\cot x = \frac{\cos\theta}{\sin\theta}$

Which of the following is equivalent to $\frac{\csc x}{\tan x + \cot x}$?

(A) $\sin x$ (B) $\cos x$

(C) $\tan x$ (D) $\cot x$

(Sol) $\frac{\csc x}{\tan x + \cot x} = \frac{\frac{1}{\sin x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\frac{1}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x}}$

$= \frac{\frac{1}{\sin x}}{\frac{1}{\sin x \cdot \cos x}} = \frac{\sin x \cdot \cos x}{\sin x} = \cos x$

Ans (B)

3 Pythagorean identities

ⓐ $\sin^2\theta + \cos^2\theta = 1$

ⓑ $\tan^2\theta + 1 = \sec^2\theta$

ⓒ $1 + \cot^2\theta = \csc^2\theta$

Which expression is equal to $(\tan^2\theta + 1)(\csc^2\theta - 1)$?

(A) $1 + \cot^2\theta$ (B) $1 - \tan^2\theta$

(C) $1 - \sec^2\theta$ (D) $1 + \sec^2\theta$

(Sol) $(\tan^2\theta + 1)(\csc^2\theta - 1) = \sec^2\theta((1 + \cot^2\theta) - 1)$

$= \sec^2\theta \cdot \cot^2\theta = \frac{1}{\cos^2\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$

$= \csc^2\theta = 1 + \cot^2\theta$

Ans (A)

4 Proofs

Addition of fractions
⇒ Common denominator

Single fraction
⇒ Reduction [← Factoring]

i) Express in terms of $\sin\theta$ or $\cos\theta$

ii) $\sin^2\theta + \cos^2\theta = 1$

Which expression is equivalent to $\frac{\sin\theta}{1 + \cos\theta} + \cot\theta$?

(A) $\csc\theta$ (B) $\sec\theta$

(C) $\tan\theta$ (D) $\cos\theta$

(Sol) $\frac{\sin\theta}{1+\cos\theta} + \cot\theta = \frac{\sin\theta}{1+\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$= \frac{\sin^2\theta + \cos\theta(1+\cos\theta)}{(1+\cos\theta)\sin\theta} = \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$

$= \frac{1+\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta} = \csc\theta$

Ans (A)

pattern drill | Ace of Base

ANSWERS ··· P. 135

1 Reciprocal

(1) Which of the following is equivalent to the expression $\dfrac{1}{\sin\theta}$?

(A) $\cos\theta$ (B) $\cot\theta$ (C) $\sec\theta$ (D) $\csc\theta$

(2) Which of the following is the multiplicative inverse of $\sec x$?

(A) $\cos x$ (B) $\csc x$ (C) $\sin x$ (D) $-\sin x$

(3) What is the value of $\tan 36°\cdot\cot 36°$?

(A) 1 (B) -1 (C) $\sqrt{3}$ (D) $-\sqrt{3}$

(4) Which of the following is equivalent to the expression $\dfrac{\sec\theta}{\csc\theta}$?

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$ (D) $\cot\theta$

2 Quotient

(1) Which of the following expressions is equivalent to $\cot x$?

(A) $\dfrac{\sin x}{\cos x}$ (B) $\dfrac{\cos x}{\sin x}$ (C) $-\dfrac{\sin x}{\cos x}$ (D) $\cos x - \sin x$

(2) Which of the following is equivalent to the expression $\dfrac{\sec\theta}{\tan\theta}$?

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) $\csc\theta$

(3) If $2\sin\theta = \cos\theta$, then what is the value of $\tan\theta$?

(A) 2 (B) $\dfrac{1}{2}$ (C) -2 (D) $-\dfrac{1}{2}$

(4) Which of the following is equivalent to the expression $\dfrac{\sin\theta - \cos^2\theta}{\sin\theta\cdot\cos\theta}$?

(A) $\tan\theta - \cot\theta$ (B) $\cot\theta - \sec\theta$ (C) $\sec\theta - \cot\theta$ (D) $\csc\theta - \tan\theta$

3 Pythagorean

(1) What is the value of $\sin^2\dfrac{\pi}{3} + \cos^2\dfrac{\pi}{3}$?

(A) 0 (B) 1 (C) 0.75 (D) 1.5

(2) Which of the following is equivalent to the expression $\dfrac{1}{1-\cos A} + \dfrac{1}{1+\cos A}$?

(A) $2\sin^2 A$ (B) $2\csc^2 A$ (C) $2\sec^2 A$ (D) $2\cot^2 A$

(3) Which of the following is equivalent to the expression $(1-\cos^2 A)(1+\tan^2 A)$?

(A) $\sin^2 A$ (B) $\cos^2 A$ (C) $\tan^2 A$ (D) $\csc^2 A$

(4) Which of the following is equivalent to the expression $(\tan x + \cot x)^2$?

(A) 1 (B) 2 (C) $\sec^2 x + \csc^2 x$ (D) $\sec^2 x\cdot\csc^2 x$

4 Proofs

(1) Which of the following is equivalent to the expression $\tan A + \cot A$?

(A) $\sin A\cdot\cos A$ (B) $\dfrac{1}{\sin A\cdot\cos A}$ (C) $2\sin A\cdot\cos A$ (D) $\dfrac{2}{\sin A\cdot\cos A}$

(2) Which of the following is equivalent to the expression $\dfrac{1-\sin x}{1-\csc x}$?

(A) $\sin x$ (B) $-\sin x$ (C) $\cos x$ (D) $-\cos x$

(3) Which of the following is equivalent to the expression $\dfrac{\cot x}{1+\cot^2 x}$?

(A) $\tan x$ (B) $\cot x$ (C) $\sin x\cdot\cos x$ (D) $\sec x$

(4) Which is equivalent to $\left(\dfrac{1}{\sin\theta}+1\right)\left(\dfrac{1}{\cos\theta}+1\right)\left(\dfrac{1}{\sin\theta}-1\right)\left(\dfrac{1}{\cos\theta}-1\right)$?

(A) 0 (B) 1 (C) $\tan^2\theta$ (D) $\cot^2\theta$

❶ (1) D (2) A (3) A (4) C ❷ (1) B (2) D (3) B (4) C ❸ (1) B (2) B (3) C (4) C ❹ (1) B (2) B (3) C (4) B

Relationships

Reciprocal	Quotient	Pythagorean
$\csc\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\sin^2\theta + \cos^2\theta = 1$
$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\tan^2\theta + 1 = \sec^2\theta$
$\cot\theta = \frac{1}{\tan\theta}$		$1 + \cot^2\theta = \csc^2\theta$

spark! Simplifying the trigonometric expressions

$\tan^2 x + 1 \xrightarrow{\text{(i)}} \sec^2 x \xrightarrow{\text{(ii)}} \frac{1}{\cos^2 x}$

$1 + \cot^2 x \rightarrow \csc^2 x \rightarrow \frac{1}{\sin^2 x}$

$\Rightarrow (\sin^2 x + \cos^2 x = 1) \Rightarrow$

$\sin^2 x \xrightarrow{\text{(iii)}} 1 - \cos^2 x \dashrightarrow f(\cos x)$

$\cos^2 x \rightarrow 1 - \sin^2 x \dashrightarrow f(\sin x)$

CAP Which of the following is equivalent to the expression

$$\frac{1}{1-\cos\theta} - \frac{\cos\theta}{1-\cos^2\theta}\ ?$$

(A) $\sin^2\theta + \cos^2\theta + \tan^2\theta$

(B) $\sin^2\theta + \cos^2\theta + \cot^2\theta$

(C) $\sin^2\theta + \cos^2\theta - \tan^2\theta$

(D) $\sin^2\theta - \cos^2\theta + \cot^2\theta$

(E) $\sin^2\theta - \cos^2\theta + \tan^2\theta$

Accent ⓐ $\sin^2\theta + \cos^2\theta = 1$ ⓑ $\tan^2\theta + 1 = \sec^2\theta$ ⓒ $1 + \cot^2\theta = \csc^2\theta$

Simple Solution

$$\frac{1}{1-\cos\theta} - \frac{\cos\theta}{1-\cos^2\theta} = \frac{1}{1-\cos\theta} - \frac{\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1+\cos\theta}{(1-\cos\theta)(1+\cos\theta)} - \frac{\cos\theta}{(1-\cos\theta)(1+\cos\theta)} = \frac{1+\cancel{\cos\theta}-\cancel{\cos\theta}}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta} = \csc^2\theta = 1 + \cot^2\theta = (\sin^2\theta + \cos^2\theta) + \cot^2\theta$$

$$= \sin^2\theta + \cos^2\theta + \cot^2\theta$$

Ans (B)

training

ANSWER ··· P. 136

(1) If $0° < x < 90°$ and $\cot x = 2$, what is the value of $\log_5 \cos x + \log_5 \tan x$?

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$

(D) $-\frac{1}{2}$ (E) -2

(2) What is the value of the expression $(\sin x - \frac{1}{\sin x})^2 + (\cos x - \frac{1}{\cos x})^2 - (\tan x + \frac{1}{\tan x})^2$?

(A) -5 (B) -3 (C) -1

(D) 1 (E) 3

KNOW-HOW

Super Model

1 Pythagorean identities ▷ $\sin^2 A + \cos^2 A = 1$

If $\dfrac{\cos\theta}{\sqrt{\tan^2\theta + 1}} - \dfrac{\sin\theta}{\sqrt{\cot^2\theta + 1}} = 1$, in which quadrant does the terminal side of θ lie?

(A) II only　　(B) III only

(C) IV only　　(D) II and IV only

$\dfrac{\cos\theta}{\sqrt{\tan^2\theta + 1}} - \dfrac{\sin\theta}{\sqrt{\cot^2\theta + 1}} = \dfrac{\cos\theta}{\sqrt{\sec^2\theta}} - \dfrac{\sin\theta}{\sqrt{\csc^2\theta}}$

$= \dfrac{\cos\theta}{|\sec\theta|} - \dfrac{\sin\theta}{|\csc\theta|} = \dfrac{\cos\theta}{\dfrac{1}{|\cos\theta|}} - \dfrac{\sin\theta}{\dfrac{1}{|\sin\theta|}}$

$= \cos\theta|\cos\theta| - \sin\theta|\sin\theta| = 1$

If $\cos\theta > 0$ and $\sin\theta < 0$ ⋯⋯ ①,

$\cos\theta \cdot \cos\theta - \sin\theta(-\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

From ①, the terminal side of θ lies in quadrant IV only.　**Ans** (C)

2 Pythagorean identities ▷ ⓐ $\sin^2\theta + \cos^2\theta = 1$　ⓑ $\tan^2\theta + 1 = \sec^2\theta$　ⓒ $1 + \cot^2\theta = \csc^2\theta$

If $\tan\theta = 7$, then what is the value of $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta}$?

(A) 10　　(B) 50

(C) 100　　(D) 200

$\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = \dfrac{1 + \sin\theta + 1 - \sin\theta}{(1 - \sin\theta)(1 + \sin\theta)}$

$= \dfrac{2}{1 - \sin^2\theta} = \dfrac{2}{\cos^2\theta} = 2 \cdot \dfrac{1}{\cos^2\theta} = 2\sec^2\theta$

$= 2(\tan^2\theta + 1) = 2(7^2 + 1) = 2(49 + 1) = 100$　**Ans** (C)

3 Proofs ▷ $\csc\alpha = \dfrac{1}{\sin\alpha}$, $\sec\alpha = \dfrac{1}{\cos\alpha}$, $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ ⇒ $\sin^2\alpha + \cos^2\alpha = 1$

Which of the following is equivalent to $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) - 2\sec\theta(\sec\theta + \tan\theta)$?

(A) $\sin\theta \cdot \cos\theta$　　(B) $\sin\theta - \cos\theta$

(C) $\dfrac{\sin\theta + 1}{\sin\theta - 1}$　　(D) $\dfrac{\cos\theta - 1}{\cos\theta + 1}$

$(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) - 2\sec\theta(\sec\theta + \tan\theta)$

$= (\csc^2\theta - \cot^2\theta) - 2 \cdot \dfrac{1}{\cos\theta}\left(\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}\right)$

$= ((1 + \cot^2\theta) - \cot^2\theta) - \dfrac{2}{\cos\theta} \cdot \dfrac{1 + \sin\theta}{\cos\theta}$

$= 1 - \dfrac{2(1 + \sin\theta)}{\cos^2\theta} = 1 - \dfrac{2(1 + \sin\theta)}{1 - \sin^2\theta}$

$= 1 - \dfrac{2(1 + \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)} = 1 - \dfrac{2}{1 - \sin\theta}$

$= \dfrac{(1 - \sin\theta) - 2}{1 - \sin\theta} = \dfrac{-\sin\theta - 1}{-\sin\theta + 1} = \dfrac{\sin\theta + 1}{\sin\theta - 1}$　**Ans** (C)

4 Simplifying ▷ $\tan x = \dfrac{\sin x}{\cos x}$ ⇒ $\sin^2 x + \cos^2 x = 1$ ⇒ $f(\sin x)$ ⇒ range

If $\cos x = 2\tan x$, then what is the value of $\csc x$?

(A) $1 + \sqrt{2}$　　(B) $\sqrt{2} - 1$

(C) $\sqrt{2}$　　(D) 4

From $\cos x = 2\tan x$, $\cos x = 2 \cdot \dfrac{\sin x}{\cos x}$

$\therefore \cos^2 x = 2\sin x$　$\therefore 1 - \sin^2 x = 2\sin x$

$\therefore \sin^2 x + 2\sin x - 1 = 0$

If $\sin x = A$, $-1 \le A \le 1$　$\therefore A^2 + 2A - 1 = 0$ (1)

$\therefore A = -1 \pm \sqrt{1^2 - 1 \cdot (-1)} = -1 \pm \sqrt{2}$

Since $-1 \le A \le 1$, $A = \sin x = -1 + \sqrt{2}$

$\therefore \csc x = \dfrac{1}{\sin x} = \dfrac{1}{\sqrt{2} - 1} = \dfrac{\sqrt{2} + 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$

$= \dfrac{\sqrt{2} + 1}{2 - 1} = 1 + \sqrt{2}$　**Ans** (A)

the melting zone

ANSWERS ··· P. 137

SHOW CASE

1 If $\tan\theta = \sqrt{\dfrac{x}{4-x}}$, which of the following expressions is equivalent to x?

(A) $2(\sec\theta + 1)(\sec\theta - 1)$ (B) $4(1 + \cos\theta)(1 - \cos\theta)$ (C) $2(1 + \sin\theta)(1 - \sin\theta)$

(D) $4(\cot\theta + 1)(\cot\theta - 1)$ (E) $2(1 + \csc\theta)(1 - \csc\theta)$

I sea! From $\tan\theta = \sqrt{\dfrac{x}{4-x}}$, $\tan^2\theta = \dfrac{x}{4-x}$

$\therefore (4-x)\tan^2\theta = x \quad \therefore 4\tan^2\theta - x\tan^2\theta = x \quad \therefore x(1 + \tan^2\theta) = 4\tan^2\theta$

$$\therefore x = \frac{4\tan^2\theta}{1 + \tan^2\theta} = \frac{4\tan^2\theta}{\sec^2\theta} = \frac{4\cdot\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{1}{\cos^2\theta}} = 4\sin^2\theta = 4(1 - \cos^2\theta)$$

$$= 4(1 + \cos\theta)(1 - \cos\theta)$$

Ans (B)

2 Which of the following expressions is equivalent to $\sec x \cdot \csc x \cdot \cot x$?

(A) $\sec^2 x - \csc^2 x - \cot^2 x$ (B) $\sec^2 x - \csc^2 x + \cot^2 x$ (C) $\sec^2 x + \csc^2 x - \cot^2 x$

(D) $\sec^2 x - \tan^2 x - \cot^2 x$ (E) $\sec^2 x - \tan^2 x + \cot^2 x$

3 If $\sin x \cdot \cos x = \dfrac{1}{5}$, what is the value of $(\sin x + \csc x)^2 + (\cos x + \sec x)^2$?

(A) 10 (B) 15 (C) 20

(D) 25 (E) 30

4 Which of the following is equivalent to the expression $\dfrac{\cos\theta}{1 - \tan\theta} + \dfrac{\sin\theta}{1 - \cot\theta} - \dfrac{\cot\theta}{\csc\theta}$?

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$

(D) $\sec\theta$ (E) $\csc\theta$

5 If $\sec x - \tan x = 0.4$, then what is the value of $\sec x + \tan x$?

(A) 1.6 (B) 2.4 (C) 2.5

(D) 20 (E) 24

6 If $3\sin^2\theta + 5\sin\theta \cdot \cos\theta - 2\cos^2\theta = 1$, then what is the value of $\tan\theta$?

(A) $\dfrac{1}{2}$ (B) 1 (C) 2

(D) $\dfrac{1}{3}$ or -2 (E) $\dfrac{1}{2}$ or -3

7 If $\pi < \theta < \dfrac{3\pi}{2}$ and $\sqrt{(1 - \cos\theta)(1 + \cos\theta)} + 1 = \dfrac{3}{\sec\theta}$, then what is the value of $\cos\theta$?

(A) 0.25 (B) 0.4 (C) −0.6

(D) −0.75 (E) −0.8

JUMP

ANSWERS ··· P. 137

1 If $\cos\theta = a\sin\theta$ and $\tan\theta = 4$, then what is the value of a?

(A) 0.25
(B) 2.5
(C) 0
(D) 1
(E) 3

2 If $0 < \theta < 90°$, which of the following is equivalent to the expression of $\log(\sec\theta + 1) + \log(\sec\theta - 1)$?

(A) $2\log\sin\theta - \log\cos\theta$
(B) $2(\log\sin\theta - \log\cos\theta)$
(C) $\log\cos\theta - \log\sin\theta$
(D) $2(\log\cos\theta - \log\sin\theta)$
(E) $\log\sin\theta + \log\cos\theta$

3 If $\sin x = \cos^2 x$, then what is the value of $\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$?

(A) 1
(B) $\sqrt{2} - 1$
(C) $\frac{1}{2}$
(D) $1 - \sqrt{2}$
(E) 2

4 If $\pi < \theta < 2\pi$ and $\sin\theta \cdot \cos\theta = \frac{12}{25}$, then what is the value of $\sec\theta + \csc\theta$?

(A) 5
(B) $\frac{7}{5}$
(C) $\frac{12}{7}$
(D) $-\frac{35}{12}$
(E) $-\frac{35}{24}$

5 Simplify;

(1) $\dfrac{\sin x \cdot \tan^2 x + \sin x}{\tan x}$

(2) $(1 - \sin^2 x)(\sec^2 x - 1) + (1 - \cos^2 x)(\csc^2 x - 1)$

(3) $\dfrac{\sin\theta \cdot \tan^2\theta + \cos\theta \cdot \tan\theta}{\sin^2\theta + \cos^2\theta + \tan^2\theta}$ $(0 < \theta < \frac{\pi}{2})$

6 (1) If A is an angle of triangle ABC such that $\tan A + \sec A = 2$, then what is the value of $\csc A$?

(2) If $\sin\alpha + \cos\alpha = \sqrt{2}$, then what is the value of $(1 + \tan\alpha)(1 + \cot\alpha)$?

7 (1) If $\dfrac{1+\tan\theta}{1-\tan\theta} = 3$, then what is the value of the expression $\dfrac{\cos\theta + 3\sin\theta}{2\cos\theta - 3\sin\theta}$?

(2) If $\tan\theta = \sqrt{2}$, then what is the value of the expression

$$\frac{1}{1+\sin\theta} - \frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta} - \frac{1}{1-\cos\theta}?$$

Math Mania!

8 If $\sin\theta + \cos\theta = \sin\theta \cdot \cos\theta$, then what is the value of $\sec\theta(\tan\theta + \cot^2\theta)$?

9 What are the degree measures of all positive acute angles x which satisfy the equation of $\sin^2 x + \cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x + \csc^2 x = 31$?

All-Round Checks

EXAMINATION

ANSWERS ··· P. 139

1 Which of the following is equivalent to $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \csc\theta)$?

(A) 1

(B) 2

(C) $\sin\theta\cos\theta$

(D) -1

(E) $\dfrac{2}{\sin\theta\cos\theta}$

2 Which of the following is equivalent to the expression $\dfrac{\tan^2\alpha\csc^2\alpha - 1}{\sec\alpha\tan^2\alpha\cos\alpha}$?

(A) 1

(B) $\sin\alpha$

(C) $\cos\alpha$

(D) $\tan\alpha$

(E) $\cot\alpha$

3 Which of the following is equivalent to the expression $2\sin^2 x + \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$?

(A) $\cos x$

(B) $\sec x$

(C) $\csc x$

(D) 1

(E) -1

4 If $\sin x + \cos x = \dfrac{1}{\sqrt{2}}$, then what is the value of $\tan^2 x + \cot^2 x$?

(A) 10

(B) 12

(C) 14

(D) 16

(E) 18

5 For all real values of θ, $\dfrac{4\cos^2\theta - 3}{a\sin\theta + b} = 1 - 2\sin\theta$. What is the sum of the integer values of a and b?

(A) 1

(B) 2

(C) 3

(D) -2

(E) -3

6 What is the value of the expression of $\cos^2\theta(1 + 2\tan\theta)(2 + \tan\theta) - 5\sin\theta\cdot\cos\theta$?

(A) 2

(B) 5

(C) 10

(D) $\dfrac{1}{2}$

(E) $\dfrac{1}{5}$

7 If $\dfrac{1 - 2\cos^2\theta}{1 - 2\sin\theta\cdot\cos\theta} = 5$, then what is the value of $\tan\theta$?

(A) $\dfrac{1}{4}$

(B) $\dfrac{2}{3}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{3}{2}$

(E) $\dfrac{4}{3}$

8 If $\sin x - \sqrt{3}\cos x = 1$, what is the negative value of $\sin x + \sqrt{3}\cos x$?

(A) -1

(B) -2

(C) -3

(D) $-\sqrt{3}$

(E) $-\dfrac{1}{\sqrt{3}}$

9 What is the value of the expression $\sin^2 36^\circ \cos^2 36^\circ + \sin^2 36^\circ \cos^4 36^\circ + \cos^6 36^\circ + \sin^2 36^\circ$?

(A) 1
(B) $\frac{1}{4}$
(C) 2
(D) $\frac{\sqrt{3}}{4}$
(E) $\frac{\sqrt{3}-1}{2}$

10 If $0^\circ < \theta < 180^\circ$, $\frac{\cos\theta}{1+\sin\theta}$ is a root of the quadratic equation $x^2 + 4x + 1 = 0$. What is the value of θ?

(A) 30°
(B) 60°
(C) 120°
(D) 135°
(E) 150°

11 If $\sin x + \sin^2 x = 1$, then what is the value of $\cos^2 x$?

(A) $\frac{-1+\sqrt{3}}{2}$
(B) $\frac{2-\sqrt{3}}{2}$
(C) $\frac{-2+\sqrt{5}}{2}$
(D) $\frac{1+\sqrt{3}}{2}$
(E) $\frac{-1+\sqrt{5}}{2}$

12 If $\frac{\sin x - \cos x}{\sin x + \cos x} = 3 - 2\sqrt{2}$, then what is the value of $\sec^2 x$?

(A) 3
(B) 5
(C) 7
(D) 9
(E) 11

13 If $\sin x + \cos x = -1$, what is the value of $\sin^{99} x + \cos^{99} x$?

(A) –1
(B) 1
(C) 0
(D) –45
(E) 45

14 For all real values of x, except 0 and 1, $f\left(\frac{x-1}{x}\right) = x$. Which of the following is equivalent to $f(\sin^2\theta)$?

(A) $\cos^2\theta$
(B) $\tan^2\theta$
(C) $\cot^2\theta$
(D) $\sec^2\theta$
(E) $\csc^2\theta$

15 For $0 < \theta < \frac{\pi}{2}$, which of the following is equivalent to $1 - \left(\frac{\sin^2\theta}{1+\cot\theta} + \frac{\cos^2\theta}{1+\tan\theta}\right)$?

(A) $\sec^2\theta$
(B) $\csc^2\theta$
(C) 0
(D) $\sin\theta \cdot \cos\theta$
(E) 1

16 If $\tan\theta = 3$, then what is the value of $\left(\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta}\right)^2$?

(A) 10
(B) 20
(C) 30
(D) 40
(E) 50

17

(1) Express $\cot\theta$ in terms of $\sec\theta$.

(2) If $\frac{\pi}{2} < \theta < \pi$, express the expression of $\sqrt{(\csc\theta - 1)(\csc\theta + 1)}$ in terms of $\tan\theta$.

(3) If $0 \le x \le 2\pi$, how many values of x satisfy the equation $2(\tan^2 x - \sec^2 x) = \tan x \cdot \csc 2x$?

(4) If $\frac{3}{2} + \log_2 \sin\theta = \log_2 \cos\theta$, then what is the value of $\log_3 \sqrt{\csc\theta}$?

18

(1) If two roots of the quadratic equation $8x^2 + 4x + C = 0$ are $\sin\theta$ and $\cos\theta$, then what is the value of C?

(2) If $\sin\theta$ and $\cos\theta$ are the two roots of the quadratic equation of $x^2 + 2kx + 2k(k-1) = 0$, then what is the value of $\frac{1}{1+\sin\theta} + \frac{1}{1+\cos\theta}$?

19

(1) If $\frac{\cot\theta + 1}{\cot\theta - 1} = 2$, then what is the value of $\sin\theta \cdot \cos\theta \cdot \tan\theta$?

(2) If $\frac{1 - 2\sin A \cdot \sin(270^\circ - A)}{\sin^2 A - \sin^2(90^\circ - A)} = \frac{4}{5}$, then what is the value of $\tan A$?

(3) If $\cot(\theta - \frac{3\pi}{2}) = -\frac{9}{10}$, then what is the value of $\frac{\sin^3\theta - \cos^3\theta}{(\sin\theta - \cos\theta)^3}$?

20

Prove;

(1) $\frac{1 - (\sin\theta - \cos\theta)^2}{2\sin^2\theta} = \frac{\tan\theta \cdot \csc^2\theta}{1 + \tan^2\theta}$

(2) $\frac{\cot\theta \cdot \cos\theta}{\cot\theta - \cos\theta} = \sec\theta + \tan\theta$

(3) $\frac{1}{\cos x + 1} - \frac{1}{\cos x - 1}$
$= 2(\sin^2 x + \cos^2 x + \cot^2 x)$

(4) $\frac{\csc A - \cos A}{\cos A(\sec A - \csc A)} = \frac{\sin^3 A - \cos^3 A}{\sin^2 A - \cos^2 A}$

21

What is the value of the expression?

(1) $\sin^2 75^\circ + \cos^2 75^\circ - \tan^2 75^\circ - \cot^2 75^\circ + \sec^2 75^\circ + \csc^2 75^\circ$

(2) $3\sin^2 75^\circ \cdot \cos^2 75^\circ + 2\cos^4 75^\circ + \sin^2 75^\circ + \sin^4 75^\circ$

22

(1) If $3\cos\theta = 8\tan\theta$, then what is the value of $\sin\theta$?

(2) If $\sin\alpha + \cos\alpha = \frac{1}{2}$, then what is the value of $\sin^3\alpha + \cos^3\alpha$?

(3) If $0^\circ < x < \pi$ and $\sin x + \cos x = \frac{1}{5}$, then what is the value of $\cos x$?

(4) If $\frac{\sec A \cdot \csc A}{\sec A - \csc A} = \frac{5}{2}$, then what is the value of $\sin^3(\frac{3\pi}{2} - A) + \cos^3(A - \frac{\pi}{2})$?

23

Prove;

(1) $\frac{\sin^3\theta}{\cos^5\theta} = \tan^5\theta + \tan^3\theta$

(2) $\frac{\tan A - \sin A}{\tan A \cdot \sin A} = \frac{\tan A \cdot \sin A}{\tan A + \sin A}$

(3) $(\sin^4 A + \cos^4 A)(\tan A + \cot A)^2$
$= \tan^2 A + \cot^2 A$

24

(1) What is the least possible value of the function $f(x) = \frac{1}{\sin^2 x} + \frac{4}{\cos^2 x}$?

(2) If $0 < \theta < \frac{\pi}{2}$, what is the smallest value of $y = (\sin\theta - \frac{1}{\sin\theta})^2 + (\cos\theta - \frac{1}{\cos\theta})^2$?

22. (1) $$\frac{\sin\frac{2}{3}\pi-\cos\frac{5}{6}\pi}{\sin\frac{17}{6}\pi-\cos\frac{8}{3}\pi}=\frac{\sin\frac{2}{3}\pi-\cos\frac{5}{6}\pi}{\sin 2\frac{5}{6}\pi-\cos 2\frac{2}{3}\pi}$$

$$=\frac{\sin\frac{2}{3}\pi-\cos\frac{5}{6}\pi}{\sin\frac{5}{6}\pi-\cos\frac{2}{3}\pi}$$

$$=\frac{\sin(\pi-\frac{\pi}{3})-\cos(\pi-\frac{\pi}{6})}{\sin(\pi-\frac{\pi}{6})-\cos(\pi-\frac{\pi}{3})}$$

$$=\frac{\sin\frac{\pi}{3}-(-\cos\frac{\pi}{6})}{\sin\frac{\pi}{6}-(-\cos\frac{\pi}{3})}=\frac{\sin\frac{\pi}{3}+\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}+\cos\frac{\pi}{3}}$$

$$=\frac{\frac{\sqrt3}{2}+\frac{\sqrt3}{2}}{\frac{1}{2}+\frac{1}{2}}=\sqrt3$$

(2) $\sin123^\circ\cdot\sin57^\circ-\cos33^\circ\cdot\cos123^\circ\cdot\cot57^\circ$

$=\sin(180^\circ-123^\circ)\cdot\sin57^\circ$

$\quad-\sin(90^\circ-33^\circ)\cdot -\cos(180^\circ-123^\circ)\cdot\cot57^\circ$

$=\sin57^\circ\cdot\sin57^\circ-\cancel{\sin57^\circ}\cdot(-\cos57^\circ)\cdot\dfrac{\cos57^\circ}{\cancel{\sin57^\circ}}$

$=\sin^2 57^\circ+\cos^2 57^\circ=1$

(3) Since $\sin\frac{5\pi}{8}=\sin(\pi-\frac{5\pi}{8})=\sin\frac{3\pi}{8}$ and

$\cos\frac{7\pi}{8}=-\cos(\pi-\frac{7\pi}{8})=-\cos\frac{\pi}{8}$,

$\sin^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}$

$=(\sin\frac{\pi}{8})^2+(\cos\frac{3\pi}{8})^2+(\sin\frac{3\pi}{8})^2+(-\cos\frac{\pi}{8})^2$

$=(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8})+(\cos^2\frac{3\pi}{8}+\sin^2\frac{3\pi}{8})$

$=1+1=2$

(4) $\sin18^\circ\cdot\cos(-72^\circ)+\sin(-108^\circ)\cdot\cos162^\circ$

$=\sin18^\circ\cdot\cos72^\circ+(-\sin108^\circ)\cdot -\cos(180^\circ-162^\circ)$

$=\sin18^\circ\cdot\sin(90^\circ-72^\circ)$

$\quad+(-\sin(180^\circ-108^\circ))\cdot(-\cos18^\circ)$

$=\sin18^\circ\cdot\sin18^\circ+(-\sin72^\circ)\cdot(-\cos18^\circ)$

$=\sin18^\circ\cdot\sin18^\circ+(-\cos(90^\circ-72^\circ))\cdot(-\cos18^\circ)$

$=\sin^2 18^\circ+\cos18^\circ\cdot\cos18^\circ$

$=\sin^2 18^\circ+\cos^2 18^\circ=1$

$$\frac{\sin432^\circ}{\cos(-162^\circ)}=\frac{\sin(432^\circ-360^\circ)}{\cos162^\circ}=\frac{\sin72^\circ}{\cos(180^\circ-162^\circ)}$$

$$=\frac{\cos(90^\circ-72^\circ)}{-\cos18^\circ}=\frac{\cos18^\circ}{-\cos18^\circ}=-1$$

$$\frac{\cot(-288^\circ)}{\tan198^\circ}=\frac{\cot\overbrace{(90^\circ\times(-3)-18^\circ)}^{\text{Quadrant I}}}{\tan\underbrace{(90^\circ\times2+18^\circ)}_{\text{Quadrant III}}}=\frac{\tan18^\circ}{\tan18^\circ}=1$$

$\therefore \sin18^\circ\cos(-72^\circ)+\sin(-108^\circ)\cos162^\circ$

$\quad+\dfrac{\sin432^\circ}{\cos(-162^\circ)}+\dfrac{\cot(-288^\circ)}{\tan198^\circ}=\cancel{1}+\cancel{(-1)}+1=1$

Ans (1) $\sqrt3$ (2) 1 (3) 2 (4) 1

23. (1) In triangle ABC, $A+B+C=180^\circ$.

$\therefore \sin(A+B)=\sin(180^\circ-C)$

$=\sin\underbrace{(90^\circ\times2-C)}_{\text{Quadrant II}}=\sin C=\cos(90^\circ-C)$

$=\cos(C-90^\circ)=0.72$

Ans 0.72

(2) $\cot(-90^\circ+A)=\cot\underbrace{(90^\circ\times(-1)+A)}_{\text{Quadrant IV}}$

$=-\tan A=2 \quad \therefore \tan A=-2$

and $\dfrac{1+2\sin(180^\circ-A)\cdot\sin(270^\circ+A)}{\cos^2(90^\circ+A)-\cos^2(-180^\circ-A)}$

$$=\frac{1+2\sin(90^\circ\times2-A)\cdot\sin(90^\circ\times3+A)}{\cos^2(90^\circ\times1+A)-\cos^2(-90^\circ\times(-2)-A)}$$

$$=\frac{1+2\sin A\cdot(-\cos A)}{(-\sin A)^2-(-\cos A)^2}$$

$$=\frac{\sin^2 A+\cos^2 A-2\sin A\cdot\cos A}{\sin^2 A-\cos^2 A}$$

$$=\frac{(\sin A-\cos A)^2}{(\sin A+\cos A)(\sin A-\cos A)}$$

$$=\frac{(\sin A-\cos A)\div\cos A}{(\sin A+\cos A)\div\cos A}=\frac{\tan A-1}{\tan A+1}$$

$$=\frac{(-2)-1}{(-2)+1}=\frac{-3}{-1}=3$$

Ans (1) 0.72 (2) 3

24. $$\sin\left(\frac{3}{2}\pi+\theta\right)\left(\frac{1+\sin\theta}{\sin\left(\frac{\pi}{2}+\theta\right)}-\frac{\cos(\pi-\theta)}{1+\sin\theta}\right)$$

$$=\sin\underbrace{\left(\frac{\pi}{2}\times3+\theta\right)}_{\text{Quadrant IV}}\cdot\left(\frac{1+\sin\theta}{\sin\underbrace{\left(\frac{\pi}{2}\times1+\theta\right)}_{\text{Quadrant II}}}-\frac{\cos\overbrace{\left(\frac{\pi}{2}\times2-\theta\right)}^{\text{Quadrant II}}}{1+\sin\theta}\right)$$

$$=-\cos\theta\left(\frac{1+\sin\theta}{\cos\theta}-\frac{-\cos\theta}{1+\sin\theta}\right)$$

$$=-\cancel{\cos\theta}\cdot\frac{(1+\sin\theta)^2+\cos^2\theta}{\cancel{\cos\theta}(1+\sin\theta)}$$

$$=-\frac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{(1+\sin\theta)}=-\frac{1+2\sin\theta+1}{1+\sin\theta}$$

$$=-\frac{2\cancel{(1+\sin\theta)}}{\cancel{(1+\sin\theta)}}=-2$$

Ans −2

6★3 relationships

P.193 pattern drill

TRIGONOMETRY

TRIGONOMETRY

❶ (1) Since the reciprocal of $\sin\theta$ is $\csc\theta$,

$$\frac{1}{\sin\theta} = \csc\theta$$

Ans (D)

(2) The multiplicative inverse(= Reciprocal) of $\sec x$ is $\cos x$.

Ans (A)

(3) $\tan 36^\circ \cdot \cot 36^\circ = \tan 36^\circ \cdot \frac{1}{\tan 36^\circ} = 1$

Ans (A)

(4) $\frac{\sec\theta}{\csc\theta} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

Ans (C)

❷ (1) $\cot x = \frac{\cos x}{\sin x}$

Ans (B)

(2) $\frac{\sec\theta}{\tan\theta} = \frac{\left(\frac{1}{\cos\theta}\right)\cdot\cos\theta}{\left(\frac{\sin\theta}{\cos\theta}\right)\cdot\cos\theta} = \frac{1}{\sin\theta} = \csc\theta$

Ans (D)

(3) From $2\sin\theta = \cos\theta$, $\frac{2\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$

$\therefore 2\tan\theta = 1 \quad \therefore \tan\theta = \frac{1}{2}$

Ans (B)

(4) $\frac{\sin\theta - \cos^2\theta}{\sin\theta\cdot\cos\theta} = \frac{\sin\theta}{\sin\theta\cdot\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cdot\cos\theta}$

$= \frac{1}{\cos\theta} - \frac{\cos\theta}{\sin\theta} = \sec\theta - \cot\theta$

Ans (C)

❸ (1) Since $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} = 1$

Ans (B)

(2) $\frac{1}{1-\cos A} + \frac{1}{1+\cos A} = \frac{1+\cos A + 1 - \cos A}{(1-\cos A)(1+\cos A)}$

$= \frac{2}{1-\cos^2 A} = \frac{2}{\sin^2 A} = 2\cdot\frac{1}{\sin^2 A} = 2\csc^2 A$

Ans (B)

(3) $(1-\cos^2 A)(1+\tan^2 A) = \sin^2 A\cdot\sec^2 A$

$= \sin^2 A\cdot\frac{1}{\cos^2 A} = \frac{\sin^2 A}{\cos^2 A} = \left(\frac{\sin A}{\cos A}\right)^2$

$= \tan^2 A$

Ans (C)

(4) $(\tan x + \cot x)^2 = \tan^2 x + 2\tan x\cdot\cot x + \cot^2 x$

$= \tan^2 x + 2\cdot 1 + \cot^2 x = (\tan^2 x + 1) + (1 + \cot^2 x)$

$= \sec^2 x + \csc^2 x$

Ans (C)

❹ (1) $\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$

$= \frac{\sin^2 A + \cos^2 A}{\cos A\cdot\sin A} = \frac{1}{\sin A\cdot\cos A}$

Ans (B)

(2) $\frac{1-\sin x}{1-\csc x} = \frac{(1-\sin x)\cdot\sin x}{\left(1-\frac{1}{\sin x}\right)\cdot\sin x}$

$= \frac{\sin x(1-\sin x)}{\sin x - 1} = \frac{-\sin x(\sin x - 1)}{\sin x - 1} = -\sin x$

Ans (B)

(3) $\frac{\cot x}{1+\cot^2 x} = \frac{\cot x}{\csc^2 x} = \frac{\left(\frac{\cos x}{\sin x}\right)\cdot\sin^2 x}{\left(\frac{1}{\sin^2 x}\right)\cdot\sin^2 x}$

$= \sin x\cdot\cos x$

Ans (C)

(4) $\left(\frac{1}{\sin\theta}+1\right)\left(\frac{1}{\cos\theta}+1\right)\left(\frac{1}{\sin\theta}-1\right)\left(\frac{1}{\cos\theta}-1\right)$

$= (\csc\theta+1)(\sec\theta+1)(\csc\theta-1)(\sec\theta-1)$

$= ((\csc\theta+1)(\csc\theta-1))((\sec\theta+1)(\sec\theta-1))$

$= (\csc^2\theta-1)(\sec^2\theta-1)$

$= ((1+\cot^2\theta)-1)((\tan^2\theta+1)-1)$

$= \cot^2\theta\cdot\tan^2\theta = (\cot\theta\cdot\tan\theta)^2 = 1^2 = 1$

Ans (B)

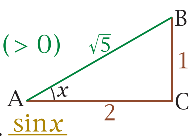
P.194 training

1. Since $\cot x = 2$, $\tan x = \frac{1}{\cot x} = \frac{1}{2}$

In right triangle ABC, AB $= \sqrt{2^2 + 1^2} = \sqrt{5}$

Since $0^\circ < x < 90^\circ$,

$\cos x = \frac{2}{\sqrt{5}}\ (>0)$ and $\sin x = \frac{1}{\sqrt{5}}\ (>0)$



$\therefore \log_5\cos x + \log_5\tan x$

$= \log_5\cos x\cdot\tan x = \log_5\cos x\cdot\frac{\sin x}{\cos x}$

$= \log_5\sin x = \log_5\frac{1}{\sqrt{5}} = \log_5 5^{-\frac{1}{2}} = -\frac{1}{2}$

Ans (D)

2. $\left(\sin x - \frac{1}{\sin x}\right)^2 + \left(\cos x - \frac{1}{\cos x}\right)^2 - \left(\tan x + \frac{1}{\tan x}\right)^2$

$= (\sin x - \csc x)^2 + (\cos x - \sec x)^2 - (\tan x + \cot x)^2$

$= (\sin^2 x - 2\sin x\cdot\csc x + \csc^2 x)$

$+ (\cos^2 x - 2\cos x\cdot\sec x + \sec^2 x)$

$- (\tan^2 x + 2\tan x\cdot\cot x + \cot^2 x)$

$= (\sin^2 x - 2\cdot 1 + \csc^2 x) + (\cos^2 x - 2\cdot 1 + \sec^2 x)$

$- (\tan^2 x + 2\cdot 1 + \cot^2 x)$

$= (\sin^2 x + \cos^2 x) + (\csc^2 x - \cot^2 x) + (\sec^2 x - \tan^2 x)$
$\quad -2-2-2$
$= 1 + ((1 + \cot^2 x) - \cot^2 x) + ((\tan^2 x + 1) - \tan^2 x) - 6$
$= (1+1+1) - 6 = -3$

Ans (B)

P.196 the melting zone

2. $\sec x \cdot \csc x \cdot \cot x = \frac{1}{\cos x} \cdot \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin^2 x}$
$= \csc^2 x = 1 + \cot^2 x = \sec^2 x - \tan^2 x + \cot^2 x$

Ans (E)

3. $(\sin x + \csc x)^2 + (\cos x + \sec x)^2$
$= (\sin^2 x + 2\sin x \cdot \csc x + \csc^2 x)$
$\quad + (\cos^2 x + 2\cos x \cdot \sec x + \sec^2 x)$
$= (\sin^2 x + 2 \cdot 1 + \csc^2 x) + (\cos^2 x + 2 \cdot 1 + \sec^2 x)$
$= (\sin^2 x + \cos^2 x) + 2 + 2 + \csc^2 x + \sec^2 x$
$= 1 + 4 + \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = 5 + \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cdot \cos^2 x}$
$= 5 + \frac{1}{(\sin x \cdot \cos x)^2} = 5 + \frac{1}{(\frac{1}{5})^2} = 5 + 25 = 30$

Ans (E)

4. $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} - \frac{\cot\theta}{\csc\theta}$

$= \left(\frac{\cos\theta}{1 - \frac{\sin\theta}{\cos\theta}}\right) \cdot \cos\theta + \left(\frac{\sin\theta}{1 - \frac{\cos\theta}{\sin\theta}}\right) \cdot \sin\theta - \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}$

$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} + \frac{\sin^2\theta}{\sin\theta - \cos\theta} - \cos\theta$

$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} - \frac{\sin^2\theta}{\cos\theta - \sin\theta} - \cos\theta$

$= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} - \cos\theta$

$= \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\cos\theta - \sin\theta} - \cos\theta$

$= (\cos\theta + \sin\theta) - \cos\theta = \sin\theta$

Ans (A)

5. If $\sec x - \tan x = 0.4$ and $\sec x + \tan x = a$,
$(\sec x - \tan x)(\sec x + \tan x) = 0.4 \times a$
$\therefore \sec^2 x - \tan^2 x = 0.4a$
$\therefore (\tan^2 x + 1) - \tan^2 x = 0.4a \quad \therefore 0.4a = 1$
$\therefore a = \frac{1}{0.4} = \frac{10}{4} = 2.5 \quad \therefore \sec x + \tan x = a = 2.5$

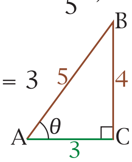
Ans (C)

6. From $3\sin^2\theta + 5\sin\theta \cdot \cos\theta - 2\cos^2\theta = 1$,
$\frac{3\sin^2\theta}{\cos^2\theta} + \frac{5\sin\theta \cdot \cos\theta}{\cos^2\theta} - \frac{2\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$3\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 5 \cdot \frac{\sin\theta}{\cos\theta} - 2 = \frac{1}{\cos^2\theta}$
$\therefore 3\tan^2\theta + 5\tan\theta - 2 = \sec^2\theta$
$\therefore 3\tan^2\theta + 5\tan\theta - 2 = \tan^2\theta + 1$
$\therefore 2\tan^2\theta + 5\tan\theta - 3 = 0$
$\therefore (2\tan\theta - 1)(\tan\theta + 3) = 0 \quad \therefore \tan\theta = \frac{1}{2}, -3$

Ans (E)

7. From $\sqrt{(1-\cos\theta)(1+\cos\theta)} + 1 = \frac{3}{\sec\theta}$,
$\sqrt{1-\cos^2\theta} + 1 = 3\cos\theta$
$\therefore \sqrt{\sin^2\theta} + 1 = 3\cos\theta \quad \therefore |\sin\theta| + 1 = 3\cos\theta$
Since $\pi < \theta < \frac{3\pi}{2}$, $\sin\theta < 0$.
$\therefore -\sin\theta + 1 = 3\cos\theta \quad \therefore (1-\sin\theta)^2 = (3\cos\theta)^2$
$\therefore 1 - 2\sin\theta + \sin^2\theta = 9\cos^2\theta$
$\therefore 1 - 2\sin\theta + \sin^2\theta = 9(1-\sin^2\theta)$
$\therefore 10\sin^2\theta - 2\sin\theta - 8 = 0 \quad \therefore 5\sin^2\theta - \sin\theta - 4 = 0$
$\therefore (5\sin\theta + 4)(\sin\theta - 1) = 0 \quad \therefore \sin\theta = -\frac{4}{5}, 1$
Since $\sin\theta < 0$, $\sin\theta = -\frac{4}{5}$



In right triangle ABC, $AC = \sqrt{5^2 - 4^2} = 3$
Since θ is an angle in quadrant III,
$\cos\theta = -\frac{3}{5} (< 0) = -0.6$

Ans (C)

P.197 JUMP

1. From $\cos\theta = a\sin\theta$, $\frac{\cos\theta}{\cos\theta} = a\frac{\sin\theta}{\cos\theta}$
$\therefore 1 = a\tan\theta \quad \therefore a = \frac{1}{\tan\theta} = \frac{1}{4} = 0.25$

Ans (A)

2. Since $0 < \theta < 90°$, $\tan\theta > 0$
$\therefore \log(\sec\theta + 1) + \log(\sec\theta - 1)$
$= \log(\sec\theta + 1)(\sec\theta - 1) = \log(\sec^2\theta - 1)$
$= \log((\tan^2\theta + 1) - 1) = \log\tan^2\theta = 2\log\tan\theta$
$= 2\log\frac{\sin\theta}{\cos\theta} = 2(\log\sin\theta - \log\cos\theta)$

Ans (B)

3. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)}$
$= \frac{2\sin x}{1-\sin^2 x} = \frac{2\sin x}{\cos^2 x}$
Since $\sin x = \cos^2 x$, $\frac{2\sin x}{\cos^2 x} = \frac{2\sin x}{\sin x} = 2$

Ans (E)

TRIGONOMETRY

4. Since $\sin\theta\cdot\cos\theta = \frac{12}{25}$ (>0),

$\sin\theta$ and $\cos\theta$ have the same sign.

From $\pi < \theta < 2\pi$, the angle θ lies in quadrant III.

Since $(\sin\theta+\cos\theta)^2 = \sin^2\theta + 2\sin\theta\cdot\cos\theta + \cos^2\theta$

$= 1 + 2\sin\theta\cdot\cos\theta = 1 + 2\cdot\frac{12}{25} = \frac{25+24}{25} = \frac{49}{25}$

$\therefore \sin\theta+\cos\theta = \pm\sqrt{\frac{49}{25}} = \pm\frac{7}{5}$

Since θ lies in quadrant III, $\sin\theta<0$ and $\cos\theta<0$

$\therefore \sin\theta+\cos\theta<0 \quad \therefore \sin\theta+\cos\theta = -\frac{7}{5}$

$\therefore \sec\theta + \csc\theta = \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = \frac{\sin\theta+\cos\theta}{\sin\theta\cdot\cos\theta}$

$= \dfrac{-\frac{7}{5}}{\frac{12}{25}} = -\frac{35}{12}$

Ans (D)

5. (1) $\dfrac{\sin x\cdot\tan^2x+\sin x}{\tan x} = \dfrac{\sin x(\tan^2x+1)}{\tan x}$

$= \dfrac{\sin x\cdot\sec^2x}{\tan x} = \dfrac{\left(\sin x\cdot\frac{1}{\cos^2x}\right)\cdot\cos^2x}{\left(\frac{\sin x}{\cos x}\right)\cdot\cos^2x}$

$= \dfrac{\sin x}{\sin x\cdot\cos x} = \dfrac{1}{\cos x} = \sec x$

(2) $(1-\sin^2x)(\sec^2x-1)+(1-\cos^2x)(\csc^2x-1)$

$= \cos^2x\cdot((\tan^2x+1)-1)$

$\quad + (1-\cos^2x)((1+\cot^2x)-1))$

$= \cos^2x\cdot\tan^2x + \sin^2x\cdot\cot^2x$

$= \cos^2x\cdot\frac{\sin^2x}{\cos^2x} + \sin^2x\cdot\frac{\cos^2x}{\sin^2x}$

$= \sin^2x+\cos^2x = 1$

(3) $\dfrac{\sin\theta\cdot\tan^2\theta+\cos\theta\cdot\tan\theta}{\sin^2\theta+\cos^2\theta+\tan^2\theta}$

$= \dfrac{\sin\theta\cdot\frac{\sin^2\theta}{\cos^2\theta}+\cos\theta\cdot\frac{\sin\theta}{\cos\theta}}{1+\tan^2\theta}$

$= \dfrac{\frac{\sin^3\theta}{\cos^2\theta}+\sin\theta}{\sec^2\theta} = \cos^2\theta\left(\frac{\sin^3\theta}{\cos^2\theta}+\sin\theta\right)$

$= \sin^3\theta+\sin\theta\cdot\cos^2\theta = \sin\theta(\sin^2\theta+\cos^2\theta)$

$= \sin\theta\cdot 1 = \sin\theta$

Ans (1) $\sec x$ (2) 1 (2) $\sin\theta$

6. (1) From $\tan A+\sec A = 2$, $\frac{\sin A}{\cos A}+\frac{1}{\cos A} = 2$

$\therefore \frac{\sin A+1}{\cos A} = 2 \quad \therefore \sin A+1 = 2\cos A$

$\therefore (\sin A+1)^2 = (2\cos A)^2$

$\therefore \sin^2A+2\sin A+1 = 4\cos^2A$

$\therefore \sin^2A+2\sin A+1 = 4(1-\sin^2A)$

$\therefore \sin^2A+2\sin A+1 = 4-4\sin^2A$

$\therefore 5\sin^2A+2\sin A-3 = 0$

$\therefore (5\sin A-3)(\sin A+1) = 0 \quad \therefore \sin A = \frac{3}{5}, -1$

Since A is an angle of triangle ABC,

$0 < A < 180^\circ \quad \therefore \sin A > 0$

$\therefore \sin A = \frac{3}{5}$ (>0) $\quad \therefore \csc A = \frac{1}{\sin A} = \frac{5}{3}$

(2) Since $\sin\alpha+\cos\alpha = \sqrt{2}$,

$(\sin\alpha+\cos\alpha)^2 = \sin^2\alpha+2\sin\alpha\cdot\cos\alpha+\cos^2\alpha$

$= 1+2\sin\alpha\cdot\cos\alpha = (\sqrt{2})^2$

$\therefore 2\sin\alpha\cdot\cos\alpha = 2-1 = 1 \quad \therefore \sin\alpha\cdot\cos\alpha = \frac{1}{2}$

$(1+\tan\alpha)(1+\cot\alpha)$

$= 1+\cot\alpha+\tan\alpha+\tan\alpha\cot\alpha$

$= 1+\frac{\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{\cos\alpha}+1 = 2+\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cdot\cos\alpha}$

$= 2+\frac{1}{\sin\alpha\cdot\cos\alpha} = 2+\dfrac{1}{\frac{1}{2}} = 2+2 = 4$

Ans (1) $\frac{5}{3}$ (2) 4

7. (1) From $\frac{1+\tan\theta}{1-\tan\theta} = 3$, $1+\tan\theta = 3(1-\tan\theta)$

$1+\tan\theta = 3-3\tan\theta \quad \therefore 4\tan\theta = 2 \quad \therefore \tan\theta = \frac{1}{2}$

$\therefore \dfrac{\frac{(\cos\theta+3\sin\theta)}{\cos\theta}}{\frac{(2\cos\theta-3\sin\theta)}{\cos\theta}} = \dfrac{1+3\frac{\sin\theta}{\cos\theta}}{2-3\frac{\sin\theta}{\cos\theta}} = \dfrac{1+3\tan\theta}{2-3\tan\theta}$

$= \dfrac{(1+3\cdot\frac{1}{2})\times 2}{(2-3\cdot\frac{1}{2})\times 2} = \dfrac{2+3}{4-3} = 5$

(2) $\frac{1}{1+\sin\theta}-\frac{1}{1+\cos\theta}+\frac{1}{1-\sin\theta}-\frac{1}{1-\cos\theta}$

$= \left(\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}\right)-\left(\frac{1}{1-\cos\theta}+\frac{1}{1-\cos\theta}\right)$

$= \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} - \frac{1-\cos\theta+1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)}$

$= \frac{2}{1-\sin^2\theta}-\frac{2}{1-\cos^2\theta} = \frac{2}{\cos^2\theta}-\frac{2}{\sin^2\theta}$

$= 2(\sec^2\theta-\csc^2\theta) = 2(\tan^2\theta+1-(1+\cot^2\theta))$

$= 2(\tan^2\theta-\cot^2\theta)) = 2\left((\sqrt{2})^2-\left(\frac{1}{\sqrt{2}}\right)^2\right)$

$= 2\left(2-\frac{1}{2}\right) = 2\cdot\frac{3}{2} = 3$

Ans (1) 5 (2) 3

8. From $\sin\theta+\cos\theta = \sin\theta\cdot\cos\theta$,

$(\sin\theta+\cos\theta)^2 = (\sin\theta\cdot\cos\theta)^2$

$\therefore \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta = (\sin\theta\cdot\cos\theta)^2$

$\therefore 1+2\sin\theta\cdot\cos\theta = (\sin\theta\cdot\cos\theta)^2$

$\therefore (\sin\theta\cdot\cos\theta)^2-2\sin\theta\cdot\cos\theta-1 = 0$

TRIGONOMETRY

If $\sin\theta\cdot\cos\theta = A,\ -\frac{1}{2} \le A \le \frac{1}{2}$

$A^2 - 2A - 1 = 0$ (-1)

$\therefore A = -(-1) \pm \sqrt{(-1)^2 - 1\cdot(-1)} = 1 \pm \sqrt{2}$

Since $-\frac{1}{2} \le A \le \frac{1}{2}$, $A = \sin\theta\cdot\cos\theta = 1 - \sqrt{2}$

$\therefore \sec\theta(\tan\theta + \cot^2\theta) = \frac{1}{\cos\theta}\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos^2\theta}{\sin^2\theta}\right)$

$= \frac{1}{\cos\theta}\cdot\frac{\sin^3\theta + \cos^3\theta}{\cos\theta\cdot\sin^2\theta} = \frac{\sin^3\theta + \cos^3\theta}{\cos^2\theta\cdot\sin^2\theta}$

$= \frac{(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cdot\cos\theta + \cos^2\theta)}{(\sin\theta\cdot\cos\theta)^2}$

$= \frac{(\sin\theta + \cos\theta)(1 - \sin\theta\cdot\cos\theta)}{(\sin\theta\cdot\cos\theta)^2}$

If $\sin\theta + \cos\theta = \sin\theta\cdot\cos\theta = A = 1 - \sqrt{2}$,

$\frac{A(1-A)}{A^2} = \frac{1-A}{A} = \frac{1 - (1 - \sqrt{2})}{1 - \sqrt{2}} = \frac{\sqrt{2}}{1 - \sqrt{2}}$

$= \frac{\sqrt{2}(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{\sqrt{2}(1 + \sqrt{2})}{1 - 2} = -\sqrt{2}(1 + \sqrt{2})$

$= -2 - \sqrt{2}$

Ans $-2 - \sqrt{2}$

9. $\sin^2 x + \cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x + \csc^2 x$

$= 1 + (\sec^2 x - 1) + (\csc^2 x - 1) + \sec^2 x + \csc^2 x$

$= 2(\sec^2 x + \csc^2 x) - 1 = 31$

$\therefore \sec^2 x + \csc^2 x = 16 \quad \therefore \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = 16$

$\therefore \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cdot\cos^2 x} = \frac{1}{(\sin x\cdot\cos x)^2} = 16$

$\therefore (\sin x\cdot\cos x)^2 = \frac{1}{16} \quad \therefore \sin x\cdot\cos x = \pm\frac{1}{4}$

Since $0 < x < 90^\circ$, $0 < \sin x\cdot\cos x < \frac{1}{4}$

$\therefore \sin x\cdot\cos x = \frac{1}{4}\ (> 0) \quad \therefore \frac{1}{2}\sin 2x = \frac{1}{4}$

$\therefore \sin 2x = \frac{1}{2} \quad \therefore 2x = 30^\circ, 150^\circ$

$\therefore x = 15^\circ, 75^\circ$ (← Positive acute angles)

Ans $15^\circ, 75^\circ$

P.198 EXAMINATION

1. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \csc\theta)$

$= \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)$

$= \frac{\cos\theta + \sin\theta + 1}{\cos\theta}\cdot\frac{\sin\theta + \cos\theta - 1}{\sin\theta}$

$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cdot\cos\theta}$

$= \frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1}{\sin\theta\cdot\cos\theta}$

$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cdot\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cdot\cos\theta} = 2$

Ans (B)

2. $\frac{\tan^2\alpha\csc^2\alpha - 1}{\sec\alpha\tan^2\alpha\cos\alpha} = \frac{\frac{\sin^2\alpha}{\cos^2\alpha}\cdot\frac{1}{\sin^2\alpha} - 1}{\frac{1}{\cos\alpha}\cdot\frac{\sin^2\alpha}{\cos^2\alpha}\cdot\cos\alpha}$

$= \frac{\left(\frac{1}{\cos^2\alpha} - 1\right)\cdot\cos^2\alpha}{\left(\frac{\sin^2\alpha}{\cos^2\alpha}\right)\cdot\cos^2\alpha} = \frac{1 - \cos^2\alpha}{\sin^2\alpha} = \frac{\sin^2\alpha}{\sin^2\alpha} = 1$

Ans (A)

3. $2\sin^2 x + \frac{1 - \tan^2 x}{1 + \tan^2 x} = 2\sin^2 x + \frac{\left(1 - \frac{\sin^2 x}{\cos^2 x}\right)\cdot\cos^2 x}{\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)\cdot\cos^2 x}$

$= 2\sin^2 x + \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = 2\sin^2 x + \frac{\cos^2 x - \sin^2 x}{1}$

$= 2\sin^2 x + \cos^2 x - \sin^2 x = \sin^2 x + \cos^2 x = 1$

Ans (D)

4. From $\sin x + \cos x = \frac{1}{\sqrt{2}}$,

$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cdot\cos x + \cos^2 x$

$= 1 + 2\sin x\cdot\cos x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\therefore 2\sin x\cdot\cos x = \frac{1}{2} - 1 = -\frac{1}{2} \quad \therefore \sin x\cdot\cos x = -\frac{1}{4}$

$\therefore \tan^2 x + \cot^2 x = (\tan x + \cot x)^2 - 2\tan x\cdot\cot x$

$= \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2 - 2\cdot 1 = \left(\frac{\sin^2 x + \cos^2 x}{\cos x\cdot\sin x}\right)^2 - 2$

$= \left(\frac{1}{\cos x\cdot\sin x}\right)^2 - 2 = \left(\frac{1}{-\frac{1}{4}}\right)^2 - 2 = (-4)^2 - 2$

$= 16 - 2 = 14$

Ans (C)

5. From the expression $\frac{4\cos^2\theta - 3}{a\sin\theta + b} = 1 - 2\sin\theta$,

$\frac{4\cos^2\theta - 3}{1 - 2\sin\theta} = a\sin\theta + b$

$\therefore \frac{4\cos^2\theta - 3}{1 - 2\sin\theta} = \frac{4(1 - \sin^2\theta) - 3}{1 - 2\sin\theta} = \frac{1 - 4\sin^2\theta}{1 - 2\sin\theta}$

$= \frac{(1 - 2\sin\theta)(1 + 2\sin\theta)}{1 - 2\sin\theta} = 1 + 2\sin\theta$

$\therefore 1 + 2\sin\theta = a\sin\theta + b \quad \therefore a = 2$ and $b = 1$

$\therefore a + b = 2 + 1 = 3$

Ans (C)

6. $\cos^2\theta(1 + 2\tan\theta)(2 + \tan\theta) - 5\sin\theta\cdot\cos\theta$

$\times\cos\theta \quad \times\cos\theta$

$= \cos^2\theta\left(1 + 2\frac{\sin\theta}{\cos\theta}\right)\left(2 + \frac{\sin\theta}{\cos\theta}\right) - 5\sin\theta\cdot\cos\theta$

$= (\cos\theta + 2\sin\theta)(2\cos\theta + \sin\theta) - 5\sin\theta\cdot\cos\theta$

$= 2\cos^2\theta + 5\sin\theta\cdot\cos\theta + 2\sin^2\theta - 5\sin\theta\cdot\cos\theta$

$= 2(\sin^2\theta + \cos^2\theta) = 2\cdot 1 = 2$

Ans (A)

TRIGONOMETRY

7. $\dfrac{1-2\cos^2\theta}{1-2\sin\theta\cdot\cos\theta}=\dfrac{\sin^2\theta+\cos^2\theta-2\cos^2\theta}{\sin^2\theta-2\sin\theta\cdot\cos\theta+\cos^2\theta}$

$=\dfrac{\sin^2\theta-\cos^2\theta}{(\sin\theta-\cos\theta)^2}=\dfrac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{(\sin\theta-\cos\theta)^2}$

$=\dfrac{(\sin\theta+\cos\theta)\div\cos\theta}{(\sin\theta-\cos\theta)\div\cos\theta}=\dfrac{\tan\theta+1}{\tan\theta-1}=5$

$\therefore \tan\theta+1=5(\tan\theta-1)\quad \therefore \tan\theta+1=5\tan\theta-5$

$\therefore 4\tan\theta=6 \quad \therefore \tan\theta=\dfrac{3}{2}$

Ans (D)

8. If the value of $\sin x+\sqrt{3}\cos x=k$,

$\begin{cases}\sin x-\sqrt{3}\cos x=1 \cdots\cdots ① \\ \sin x+\sqrt{3}\cos x=k \cdots\cdots ②\end{cases}$

From ②−①, $2\sqrt{3}\cos x=k-1\quad \therefore \cos x=\dfrac{k-1}{2\sqrt{3}}\cdots\cdots ③$

From ③→①, $\sin x-\sqrt{3}\cdot\dfrac{k-1}{2\sqrt{3}}=1$

$\therefore \sin x-\dfrac{k-1}{2}=1\quad \therefore \sin x=1+\dfrac{k-1}{2}=\dfrac{k+1}{2}\cdots\cdots ④$

From ③, ④ and $\sin^2x+\cos^2x=1$,

$\left(\dfrac{k+1}{2}\right)^2+\left(\dfrac{k-1}{2\sqrt{3}}\right)^2=\dfrac{(k+1)^2}{4}+\dfrac{(k-1)^2}{12}=1$

$\therefore \dfrac{3(k+1)^2+(k-1)^2}{12}=1\quad \therefore 3(k+1)^2+(k-1)^2=12$

$\therefore 3(k^2+2k+1)+(k^2-2k+1)=12$

$\therefore 4k^2+4k+4=12\quad \therefore 4k^2+4k-8=0$

$\therefore k^2+k-2=0\quad \therefore (k+2)(k-1)=0\quad \therefore k=-2,\ 1$

$\therefore \sin x+\sqrt{3}\cos x=k=-2$ or 1

$\therefore$ The negative value of $\sin x+\sqrt{3}\cos x$ is -2.

Ans (B)

9. If $\theta=36^\circ$,

$\sin^2 36^\circ\cos^2 36^\circ+\sin^2 36^\circ\cos^4 36^\circ+\cos^6 36^\circ+\sin^2 36^\circ$

$=\sin^2\theta\cos^2\theta+\sin^2\theta\cos^4\theta+\cos^6\theta+\sin^2\theta$

$=(1-\cos^2\theta)\cos^2\theta+(1-\cos^2\theta)\cos^4\theta+\cos^6\theta$

$\quad+(1-\cos^2\theta)$

$=\cos^2\theta-\cos^4\theta+\cos^4\theta-\cos^6\theta+\cos^6\theta+1-\cos^2\theta$

$=1$

Ans (A)

10. Since $\dfrac{\cos\theta}{1+\sin\theta}$ is a root of $x^2+4x+1=0$,

$x=\dfrac{\cos\theta}{1+\sin\theta}\quad \therefore \dfrac{1}{x}=\dfrac{1+\sin\theta}{\cos\theta}$

$\therefore x+\dfrac{1}{x}=\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}$

$=\dfrac{\cos^2\theta+(1+\sin\theta)^2}{\cos\theta(1+\sin\theta)}$

$=\dfrac{\cos^2\theta+(1+2\sin\theta+\sin^2\theta)}{\cos\theta(1+\sin\theta)}$

$=\dfrac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\dfrac{2}{\cos\theta}\cdots\cdots ①$

From the quadratic equation $x^2+4x+1=0$,

$x+4+\dfrac{1}{x}=0\quad \therefore x+\dfrac{1}{x}=-4$

From ①, $\dfrac{2}{\cos\theta}=-4\quad \therefore \cos\theta=\dfrac{2}{-4}=-\dfrac{1}{2}$

Since $0^\circ<\theta<180^\circ$, $\theta=120^\circ$

Ans (C)

11. From $\sin x+\sin^2x=1$,

$\sin x=1-\sin^2x\quad \therefore \sin x=\cos^2x\,(\geq 0)$

If $\sin x=t$, $t\geq 0$

From $\sin x+\sin^2x=1$, $t^2+t-1=0$

$\therefore t=\dfrac{-1\pm\sqrt{1^2-4\cdot 1\cdot(-1)}}{2}=\dfrac{-1\pm\sqrt{5}}{2}$

Since $t\geq 0$, $t=\dfrac{-1+\sqrt{5}}{2}$

$\therefore t=\sin x=\cos^2x=\dfrac{-1+\sqrt{5}}{2}$

Ans (E)

12. From $\dfrac{\sin x-\cos x}{\sin x+\cos x}=3-2\sqrt{2}$,

$\dfrac{\dfrac{(\sin x-\cos x)}{\cos x}}{\dfrac{(\sin x+\cos x)}{\cos x}}=\dfrac{\tan x-1}{\tan x+1}=3-2\sqrt{2}$

$\therefore \tan x-1=(3-2\sqrt{2})(\tan x+1)$

$\therefore \tan x-1=(3-2\sqrt{2})\tan x+3-2\sqrt{2}$

$\therefore (2-2\sqrt{2})\tan x=-4+2\sqrt{2}$

$\therefore \tan x=\dfrac{2(-2+\sqrt{2})}{2(1-\sqrt{2})}=\dfrac{\sqrt{2}(1-\sqrt{2})}{1-\sqrt{2}}=\sqrt{2}$

$\therefore \sec^2x=\tan^2x+1=(\sqrt{2})^2+1=2+1=3$

Ans (A)

13. From $\sin x+\cos x=-1\cdots\cdots ①$,

$(\sin x+\cos x)^2=(-1)^2$

$\therefore \sin^2x+2\sin x\cos x+\cos^2x=1$

$\therefore 1+2\sin x\cos x=1\quad \therefore 2\sin x\cos x=0$

$\therefore \sin x=0$ or $\cos x=0$

i) $\sin x=0 \Rightarrow$ From ①, $\cos x=-1$

$\therefore \sin^{99}x+\cos^{99}x=0+(-1)^{99}=-1$

ii) $\cos x=0 \Rightarrow$ From ①, $\sin x=-1$

$\therefore \sin^{99}x+\cos^{99}x=(-1)^{99}+0=-1$

i) and ii), $\sin^{99}x+\cos^{99}x=-1$

Ans (A)

14. From $f\left(\dfrac{x-1}{x}\right)=x$, if $\dfrac{x-1}{x}=A$, $x-1=Ax$

$\therefore x-Ax=1\quad \therefore x(1-A)=1\quad \therefore x=\dfrac{1}{1-A}$

$\therefore f(A)=\dfrac{1}{1-A}$

$\therefore f(\sin^2\theta)=\dfrac{1}{1-\sin^2\theta}=\dfrac{1}{\cos^2\theta}=\sec^2\theta$

Ans (D)

TRIGONOMETRY

15. $1-\left(\dfrac{\sin^2\theta}{1+\cot\theta}+\dfrac{\cos^2\theta}{1+\tan\theta}\right)$

$=1-\left(\dfrac{(\sin^2\theta)\cdot\sin\theta}{\left(1+\frac{\cos\theta}{\sin\theta}\right)\cdot\sin\theta}+\dfrac{(\cos^2\theta)\cdot\cos\theta}{\left(1+\frac{\sin\theta}{\cos\theta}\right)\cdot\cos\theta}\right)$

$=1-\left(\dfrac{\sin^3\theta}{\sin\theta+\cos\theta}+\dfrac{\cos^3\theta}{\cos\theta+\sin\theta}\right)$

$=1-\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=\dfrac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}-\dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cdot\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(1-(1-\sin\theta\cdot\cos\theta))}{\sin\theta+\cos\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(\sin\theta\cdot\cos\theta)}{\sin\theta+\cos\theta}=\sin\theta\cdot\cos\theta$

Ans (D)

16. $\left(\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}\right)^2$

$=\left(\dfrac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}\right)^2$

$=\left(\dfrac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}\right)^2$

$=\left(\dfrac{1+1+2\sin\theta}{(1+\sin\theta)\cos\theta}\right)^2=\left(\dfrac{2(1+\sin\theta)}{(1+\sin\theta)\cdot\cos\theta}\right)^2$

$=\left(\dfrac{2}{\cos\theta}\right)^2=\dfrac{4}{\cos^2\theta}=4\sec^2\theta=4(\tan^2\theta+1)$

$=4(3^2+1)=4\cdot10=40$

Ans (D)

17. (1) Since $\tan^2\theta+1=\sec^2\theta$,

$\tan^2\theta=\sec^2\theta-1\quad\therefore\tan\theta=\pm\sqrt{\sec^2\theta-1}$

Since $\cot\theta=\dfrac{1}{\tan\theta}$, $\cot\theta=\pm\dfrac{1}{\sqrt{\sec^2\theta-1}}$

(2) $\sqrt{(\csc\theta-1)(\csc\theta+1)}=\sqrt{\csc^2\theta-1}$

$=\sqrt{(1+\cot^2\theta)-1}=\sqrt{\cot^2\theta}=|\cot\theta|$

Since $\dfrac{\pi}{2}<\theta<\pi$, $\cot\theta<0$

$\therefore|\cot\theta|=-\cot\theta=-\dfrac{1}{\tan\theta}$

(3) From the equation

$2(\tan^2x-\sec^2x)=\tan^2x\cdot\csc^2x$,

$2(\tan^2x-(\tan^2x+1))=\dfrac{\sin^2x}{\cos^2x}\cdot\dfrac{1}{\sin^2x}$

$\therefore 2(-1)=\dfrac{1}{\cos^2x}\quad\therefore\cos^2x=-\dfrac{1}{2}$

Since $\cos^2x\geq0$, $\cos^2x\neq-\dfrac{1}{2}$

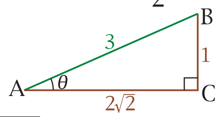
There are no values of x that satisfy the equation.

(4) From $\dfrac{3}{2}+\log_2\sin\theta=\log_2\cos\theta$,

$\log_2\sin\theta-\log_2\cos\theta=-\dfrac{3}{2}$

$\therefore\log_2\dfrac{\sin\theta}{\cos\theta}=-\dfrac{3}{2}\quad\therefore\log_2\tan\theta=-\dfrac{3}{2}$

$\therefore\tan\theta=2^{-\frac{3}{2}}=\dfrac{1}{2\sqrt2}$



In right triangle ABC,

$AB=\sqrt{(2\sqrt2)^2+1^2}=\sqrt{8+1}=3$

$\therefore\sin\theta=\dfrac{1}{3}\quad\therefore\csc\theta=3$

$\therefore\log_3\sqrt{\csc\theta}=\log_3\sqrt3=\dfrac{1}{2}$

Ans (1) $\pm\dfrac{1}{\sqrt{\sec^2\theta-1}}$ (2) $-\dfrac{1}{\tan\theta}$

(3) None(0) (4) $\dfrac{1}{2}$

18. (1) Since the quadratic equation $8x^2+4x+C=0$ has two roots of $\sin\theta$ and $\cos\theta$,

$\begin{cases}\sin\theta+\cos\theta=-\dfrac{4}{8}=-\dfrac{1}{2}\ \cdots\cdots①\\ \sin\theta\cdot\cos\theta=\dfrac{C}{8}\ \cdots\cdots②\end{cases}$

From ①, $(\sin\theta+\cos\theta)^2=\left(-\dfrac{1}{2}\right)^2$

$\therefore\sin^2\theta+2\sin\theta\cdot\cos\theta+\cos^2\theta=\dfrac{1}{4}$

$\therefore 1+2\sin\theta\cdot\cos\theta=\dfrac{1}{4}\quad\therefore 2\sin\theta\cdot\cos\theta=-\dfrac{3}{4}$

$\therefore\sin\theta\cdot\cos\theta=-\dfrac{3}{8}\ \cdots\cdots③$

From ③ → ②, $-\dfrac{3}{8}=\dfrac{C}{8}\quad\therefore C=-3$

(2) From the quadratic equation $x^2+2kx+2k(k-1)=0$,

$\begin{cases}\sin\theta+\cos\theta=-2k\ \cdots\cdots①\\ \sin\theta\cdot\cos\theta=2k(k-1)\ \cdots\cdots②\end{cases}$

From ①, $(\sin\theta+\cos\theta)^2=(-2k)^2$

$\therefore\sin^2\theta+2\sin\theta\cdot\cos\theta+\cos^2\theta$

$=1+2\sin\theta\cdot\cos\theta=4k^2$

$\therefore 2\sin\theta\cdot\cos\theta=4k^2-1\cdots\cdots③$

From ② → ③, $2\cdot2k(k-1)=4k^2-1$

$\therefore 4k^2-4k=4k^2-1\quad\therefore -4k=-1\quad\therefore k=\dfrac{1}{4}\cdots\cdots④$

From ④ → ①, $\sin\theta+\cos\theta=-2\cdot\dfrac{1}{4}=-\dfrac{1}{2}$

From ④ → ②, $\sin\theta\cdot\cos\theta=2\cdot\dfrac{1}{4}\left(\dfrac{1}{4}-1\right)$

$=\dfrac{1}{2}\cdot\left(-\dfrac{3}{4}\right)=-\dfrac{3}{8}$

$\therefore\dfrac{1}{1+\sin\theta}+\dfrac{1}{1+\cos\theta}=\dfrac{1+\cos\theta+1+\sin\theta}{(1+\sin\theta)(1+\cos\theta)}$

$$= \frac{2 + (\sin\theta + \cos\theta)}{1 + (\sin\theta + \cos\theta) + \sin\theta \cdot \cos\theta}$$

$$= \frac{\left(2 + \left(-\frac{1}{2}\right)\right)\cdot 8}{\left(1 + \left(-\frac{1}{2}\right) + \left(-\frac{3}{8}\right)\right)\cdot 8} = \frac{16 - 4}{8 - 4 - 3} = 12$$

Ans (1) -3 (2) 12

19. (1) From $\frac{\cot\theta + 1}{\cot\theta - 1} = 2$, $\cot\theta + 1 = 2(\cot\theta - 1)$

$\therefore \cot\theta + 1 = 2\cot\theta - 2 \quad \therefore \cot\theta = 3$

$\therefore \sin\theta \cdot \cos\theta \cdot \tan\theta = \sin\theta \cdot \cos\theta \cdot \frac{\sin\theta}{\cos\theta}$

$= \sin^2\theta = \frac{1}{\csc^2\theta} = \frac{1}{1 + \cot^2\theta} = \frac{1}{1 + 3^2}$

$= \frac{1}{10}$

(2) Since $\sin(270° - A) = \sin(90° \times 3 - A)$ (Quadrant III)

$= -\cos A = -\cos A$

and $\sin(90° - A) = \cos A$,

$$\frac{1 - 2\sin A \cdot \sin(270° - A)}{\sin^2 A - \sin^2(90° - A)}$$

$$= \frac{1 - 2\sin A \cdot (-\cos A)}{\sin^2 A - \cos^2 A} = \frac{1 + 2\sin A \cdot \cos A}{\sin^2 A - \cos^2 A}$$

$$= \frac{\sin^2 A + \cos^2 A + 2\sin A \cdot \cos A}{\sin^2 A - \cos^2 A}$$

$$= \frac{(\sin A + \cos A)^2}{(\sin A + \cos A)(\sin A - \cos A)}$$

$$= \frac{\frac{(\sin A + \cos A)}{\cos A}}{\frac{(\sin A - \cos A)}{\cos A}} = \frac{\tan A + 1}{\tan A - 1} = \frac{4}{5}$$

$\therefore 5(\tan A + 1) = 4(\tan A - 1)$

$\therefore 5\tan A + 5 = 4\tan A - 4 \quad \therefore \tan A = -9$

(3) Since $\cot\left(\theta - \frac{3\pi}{2}\right) = \cot\left(\frac{\pi}{2} \times (-3) + \theta\right)$ (Quadrant II)

$= -\tan\theta = -\frac{9}{10} \quad \therefore \tan\theta = \frac{9}{10}$

$$\therefore \frac{\sin^3\theta - \cos^3\theta}{(\sin\theta - \cos\theta)^3}$$

$$= \frac{(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta \cdot \cos\theta + \cos^2\theta)}{(\sin\theta - \cos\theta)(\sin\theta - \cos\theta)^2}$$

$$= \frac{\frac{(\sin^2\theta + \sin\theta \cdot \cos\theta + \cos^2\theta)}{\cos^2\theta}}{\frac{(\sin^2\theta - 2\sin\theta \cdot \cos\theta + \cos^2\theta)}{\cos^2\theta}}$$

$$= \frac{\tan^2\theta + \tan\theta + 1}{\tan^2\theta - 2\tan\theta + 1}$$

$$= \frac{\left(\left(\frac{9}{10}\right)^2 + \frac{9}{10} + 1\right)\cdot 100}{\left(\left(\frac{9}{10}\right)^2 + 2\left(\frac{9}{10}\right) + 1\right)\cdot 100}$$

$$= \frac{81 + 90 + 100}{81 - 180 + 100} = 271$$

Ans (1) $\frac{1}{10}$ (2) -9 (3) 271

20. (1) LHS $= \frac{1 - (\sin\theta - \cos\theta)^2}{2\sin^2\theta}$

$$= \frac{1 - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)}{2\sin^2\theta}$$

$$= \frac{1 - (1 - 2\sin\theta\cos\theta)}{2\sin^2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

RHS $= \frac{\tan\theta \cdot \csc^2\theta}{1 + \tan^2\theta} = \frac{\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin^2\theta}}{\sec^2\theta}$

$$= \frac{\left(\frac{1}{\sin\theta \cdot \cos\theta}\right)\cdot \cos^2\theta}{\left(\frac{1}{\cos^2\theta}\right)\cdot \cos^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$\therefore$ LHS = RHS

(2) LHS $= \frac{\cot\theta \cdot \cos\theta}{\cot\theta - \cos\theta} = \frac{\left(\frac{\cos\theta}{\sin\theta} \cdot \cos\theta\right)\cdot \sin\theta}{\left(\frac{\cos\theta}{\sin\theta} - \cos\theta\right)\cdot \sin\theta}$

$$= \frac{\cos^2\theta}{\cos\theta - \sin\theta \cdot \cos\theta} = \frac{\cos^2\theta}{\cos\theta(1 - \sin\theta)}$$

$$= \frac{\cos\theta}{1 - \sin\theta} = \frac{\cos\theta(1 + \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)}$$

$$= \frac{\cos\theta(1 + \sin\theta)}{1 - \sin^2\theta} = \frac{\cos\theta(1 + \sin\theta)}{\cos^2\theta}$$

$$= \frac{1 + \sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$= \sec\theta + \tan\theta =$ RHS

(3) LHS $= \frac{1}{\cos x + 1} - \frac{1}{\cos x - 1}$

$$= \frac{\cos x - 1 - (\cos x + 1)}{(\cos x + 1)(\cos x - 1)}$$

$$= \frac{\cos x - 1 - \cos x - 1}{\cos^2 x - 1} = \frac{-2}{\cos^2 x - 1}$$

$$= \frac{2}{1 - \cos^2 x} = \frac{2}{\sin^2 x} = 2\csc^2 x$$

$= 2(1 + \cot^2 x)$

$= 2(\sin^2 x + \cos^2 x + \cot^2 x) =$ RHS

(4) LHS $= \frac{\csc A - \cos A}{\cos A(\sec A - \csc A)}$

$$= \frac{\left(\frac{1}{\sin A} - \cos A\right)\sin A \cdot \cos A}{\cos A\left(\frac{1}{\cos A} - \frac{1}{\sin A}\right)\sin A \cdot \cos A}$$

$$= \frac{\cos A - \sin A \cdot \cos^2 A}{\cos A(\sin A - \cos A)} = \frac{\cos A(1 - \sin A \cdot \cos A)}{\cos A(\sin A - \cos A)}$$

$$= \frac{1 - \sin A \cdot \cos A}{\sin A - \cos A}$$

$$= \frac{(\sin^2 A - \sin A \cdot \cos A + \cos^2 A)(\sin A + \cos A)}{(\sin A - \cos A)(\sin A + \cos A)}$$

$$= \frac{\sin^3 A + \cos^3 A}{\sin^2 A - \cos^2 A} = \text{RHS}$$

Ans (1) Q.E.D. (2) Q.E.D. (3) Q.E.D. (4) Q.E.D.

21. (1) $\sin^2 75° + \cos^2 75° - \tan^2 75° - \cot^2 75°$

$+ \sec^2 75° + \csc^2 75°$

$= 1 - \tan^2 75° - \cot^2 75° + (\tan^2 75° + 1)$

$+ (1 + \cot^2 75°)$

$= 1 - \tan^2 75° - \cot^2 75° + \tan^2 75° + 1$

$+ 1 + \cot^2 75° = 1 + 1 + 1 = 3$

(2) Since $\sin^2 75° + \cos^2 75° = 1$,

if $\sin^2 75° = A$, $\cos^2 75° = 1 - A$

$\therefore 3\sin^2 75° \cdot \cos^2 75° + 2\cos^4 75° + \sin^2 75°$

$+ \sin^4 75°$

$= 3\sin^2 75° \cdot \cos^2 75° + 2(\cos^2 75°)^2 + \sin^2 75°$

$+ (\sin^2 75°)^2$

$= 3A(1 - A) + 2(1 - A)^2 + A + A^2$

$= 3A - 3A^2 + 2(1 - 2A + A^2) + A + A^2$

$= 3A - 3A^2 + 2 - 4A + 2A^2 + A + A^2 = 2$

Ans (1) 3 (2) 2

22. (1) From $3\cos\theta = 8\tan\theta$, $3\cos\theta = 8\,\frac{\sin\theta}{\cos\theta}$

$\therefore 3\cos^2\theta = 8\sin\theta \quad \therefore 3(1 - \sin^2\theta) = 8\sin\theta$

$\therefore 3\sin^2\theta + 8\sin\theta - 3 = 0$

$\therefore (3\sin\theta - 1)(\sin\theta + 3) = 0 \quad \therefore \sin\theta = \frac{1}{3}, -3$

Since $-1 \le \sin\theta \le 1$, $\sin\theta = \frac{1}{3}$

(2) From $\sin\alpha + \cos\alpha = \frac{1}{2}$ ······①,

$(\sin\alpha + \cos\alpha)^2 = \sin^2\alpha + 2\sin\alpha \cdot \cos\alpha + \cos^2\alpha$

$= 1 + 2\sin\alpha \cdot \cos\alpha = (\frac{1}{2})^2$

$\therefore 2\sin\alpha \cdot \cos\alpha = \frac{1}{4} - 1 = -\frac{3}{4}$

$\therefore \sin\alpha \cdot \cos\alpha = -\frac{3}{8}$ ······ ②

From ① and ②, $\sin^3\alpha + \cos^3\alpha$

$= (\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha \cdot \cos\alpha + \cos^2\alpha)$

$= \frac{1}{2} \cdot \left(1 - \left(-\frac{3}{8}\right)\right) = \frac{1}{2} \cdot \frac{11}{8} = \frac{11}{16}$

(3) From $\sin x + \cos x = \frac{1}{5}$, $\sin x = \frac{1}{5} - \cos x$

Since $0° < x < \pi$,

$0 < \sin x \le 1 \quad \therefore 0 < \frac{1}{5} - \cos x \le 1$

$\therefore -\frac{1}{5} < -\cos x \le \frac{4}{5} \quad \therefore -\frac{4}{5} \le \cos x < \frac{1}{5}$

$\therefore \sin^2 x = (\frac{1}{5} - \cos x)^2$

$\therefore 1 - \cos^2 x = \frac{1}{25} - \frac{2}{5}\cos x + \cos^2 x$

$\therefore 2\cos^2 x - \frac{2}{5}\cos x - \frac{24}{25} = 0$

$\therefore 50\cos^2 x - 10\cos x - 24 = 0$

$\therefore 25\cos^2 x - 5\cos x - 12 = 0$

$\therefore (5\cos x - 4)(5\cos x + 3) = 0$

$\therefore \cos x = \frac{4}{5}, -\frac{3}{5}$

Since $-\frac{4}{5} \le \cos x < \frac{1}{5}$, $\cos x = -\frac{3}{5}$

(4) From $\frac{\sec A \cdot \csc A}{\sec A - \csc A} = \frac{5}{2}$, $\frac{\sec A - \csc A}{\sec A \cdot \csc A} = \frac{2}{5}$

$\therefore \frac{1}{\csc A} - \frac{1}{\sec A} = \frac{2}{5} \quad \therefore \sin A - \cos A = \frac{2}{5}$

$\therefore (\sin A - \cos A)^2 = (\frac{2}{5})^2$

$\therefore \sin^2 A - 2\sin A \cdot \cos A + \cos^2 A = \frac{4}{25}$

$\therefore 1 - 2\sin A \cdot \cos A = \frac{4}{25}$

$\therefore 2\sin A \cdot \cos A = 1 - \frac{4}{25} = \frac{21}{25}$

$\therefore \sin A \cdot \cos A = \frac{21}{50}$

And $\underbrace{\sin^3(\frac{3\pi}{2} - A)}_{\text{Quadrant III}} + \underbrace{\cos^3(A - \frac{\pi}{2})}_{\text{Quadrant IV}}$

$= (-\cos A)^3 + (\sin A)^3 = \sin^3 A - \cos^3 A$

Since $(\sin A - \cos A)^3$

$= \sin^3 A - \cos^3 A - 3\sin A \cdot \cos A(\sin A - \cos A)$,

$(\frac{2}{5})^3 = \sin^3 A - \cos^3 A - 3 \cdot \frac{21}{50_{25}} \cdot \frac{2}{5}$

$\therefore \sin^3 A - \cos^3 A = \frac{8}{125} + \frac{63}{125} = \frac{71}{125}$

Ans (1) $\frac{1}{3}$ (2) $\frac{11}{16}$ (3) $-\frac{3}{5}$ (4) $\frac{71}{125}$

23. (1) LHS $= \frac{\sin^3\theta}{\cos^5\theta} = \frac{\sin^3\theta}{\cos^3\theta} \cdot \frac{1}{\cos^2\theta}$

$= \left(\frac{\sin\theta}{\cos\theta}\right)^3 \cdot \left(\frac{1}{\cos\theta}\right)^2 = \tan^3\theta \cdot \sec^2\theta$

$= \tan^3\theta \cdot (\tan^2\theta + 1) = \tan^5\theta + \tan^3\theta = \text{RHS}$

(2) From $\boxed{\frac{\tan A - \sin A}{\tan A \cdot \sin A} = \frac{\tan A \cdot \sin A}{\tan A + \sin A}}$,

TRIGONOMETRY

$$\frac{\left(\frac{\sin A}{\cos A} - \sin A\right)\cos A}{\left(\frac{\sin A}{\cos A} \cdot \sin A\right)\cos A} = \frac{\left(\frac{\sin A}{\cos A} \cdot \sin A\right)\cos A}{\left(\frac{\sin A}{\cos A} + \sin A\right)\cos A}$$

$$\therefore \frac{\sin A - \sin A \cdot \cos A}{\sin^2 A} = \frac{\sin^2 A}{\sin A + \sin A \cdot \cos A}$$

$$\therefore \frac{\sin A(1 - \cos A)}{\sin^2 A} = \frac{\sin^2 A}{\sin A(1 + \cos A)}$$

$$\therefore \boxed{\frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}}$$

$$\therefore \text{LHS} = \frac{1 - \cos A}{\sin A} = \frac{(1 - \cos A)(1 + \cos A)}{\sin A(1 + \cos A)}$$

$$= \frac{1 - \cos^2 A}{\sin A(1 + \cos A)} = \frac{\sin^2 A}{\sin A(1 + \cos A)}$$

$$= \frac{\sin A}{1 + \cos A} = \text{RHS}$$

(3) $\text{LHS} = (\sin^4 A + \cos^4 A)(\tan A + \cot A)^2$

$$= (\sin^4 A + \cos^4 A)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)^2$$

$$= (\sin^4 A + \cos^4 A)\left(\frac{\sin^2 A + \cos^2 A}{\sin A \cdot \cos A}\right)^2$$

$$= (\sin^4 A + \cos^4 A)\left(\frac{1}{\sin A \cdot \cos A}\right)^2$$

$$= \frac{\sin^4 A + \cos^4 A}{\sin^2 A \cdot \cos^2 A} = \left(\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A}\right)$$

$$= \tan^2 A + \cot^2 A = \text{RHS}$$

Ans (1) Q.E.D. (2) Q.E.D. (3) Q.E.D.

24. (1) $f(x) = \frac{1}{\sin^2 x} + \frac{4}{\cos^2 x} = \csc^2 x + 4\sec^2 x$

$$= (1 + \cot^2 x) + 4(\tan^2 x + 1)$$

$$= \cot^2 x + 4\tan^2 x + 5$$

Since $\cot^2 x \geq 0$ and $\tan^2 x \geq 0$,

$$\frac{\cot^2 x + 4\tan^2 x}{2} \geq \sqrt{\cot^2 x \cdot 4\tan^2 x}$$

$$\therefore \cot^2 x + 4\tan^2 x \geq 2\sqrt{4(\cot x \cdot \tan x)^2} = 2\sqrt{4} = 4$$

$$\therefore \cot^2 x + 4\tan^2 x + 5 \geq 4 + 5 = 9$$

$$\therefore f(x) = \frac{1}{\sin^2 x} + \frac{4}{\cos^2 x} \geq 9$$

$\therefore$ The least value of the function $f(x)$ is 9.

(2) $y = \left(\sin\theta - \frac{1}{\sin\theta}\right)^2 + \left(\cos\theta - \frac{1}{\cos\theta}\right)^2$

$$= \sin^2\theta - 2 + \frac{1}{\sin^2\theta} + \cos^2\theta - 2 + \frac{1}{\cos^2\theta}$$

$$= \sin^2\theta + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} - 2 - 2$$

$$= 1 + \frac{\text{con}^2\theta + \sin^2\theta}{\sin^2\theta \cdot \cos^2\theta} - 4 = \frac{1}{\sin^2\theta \cdot \cos^2\theta} - 3$$

Since $\sin\theta > 0$ and $\cos\theta > 0$,

$$\frac{\sin^2\theta + \cos^2\theta}{2} \geq \sqrt{\sin^2\theta \cdot \cos^2\theta}$$

$$\therefore \sin^2\theta + \cos^2\theta \geq 2\sqrt{(\sin\theta \cdot \cos\theta)^2}$$

$$\therefore 1 \geq 2\sin\theta \cdot \cos\theta \quad \therefore \sin\theta \cdot \cos\theta \leq \frac{1}{2}$$

$$\therefore (\sin\theta \cdot \cos\theta)^2 \leq \left(\frac{1}{2}\right)^2 \quad \therefore \sin^2\theta \cdot \cos^2\theta \leq \frac{1}{4}$$

$$\therefore \frac{1}{\sin^2\theta \cdot \cos^2\theta} \geq 4$$

$$\therefore \frac{1}{\sin^2\theta \cdot \cos^2\theta} - 3 \geq 4 - 3 = 1$$

$$\therefore y = \left(\sin\theta - \frac{1}{\sin\theta}\right)^2 + \left(\cos\theta - \frac{1}{\cos\theta}\right)^2 \geq 1$$

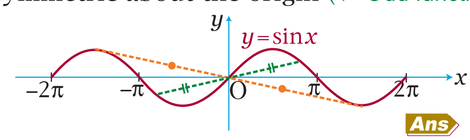
$\therefore$ The minimum value of the function is 1.

Ans (1) 9 (2) 1

6★4 trigonometric graphs

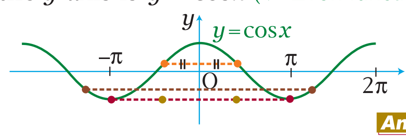
P.203 pattern drill

❶ (1) In the figure, the graph of $y = \sin x$ is symmetric about the origin (← Odd function).



Ans (C)

(2) Since the graph of $y = \cos x$ is symmetric about the y-axis, the reflection of $y = \cos x$ to the y-axis is $y = \cos x$ (← Even function).

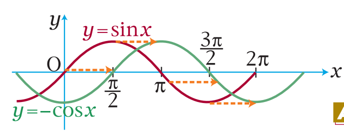


Ans (B)

(3) $y = \sin x \xrightarrow[\text{Shift right } \frac{\pi}{2} \text{ units}]{T_{\frac{\pi}{2},0}} y = \sin\left(x - \frac{\pi}{2}\right)$

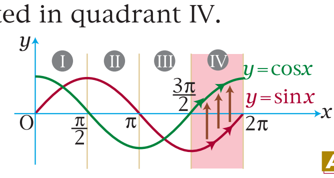
$= \sin\left(\frac{\pi}{2} \times (-1) + x\right) = \circleddash\cos x \quad \therefore y = -\cos x$

Quadrant IV



Ans (C)

(4) In the figure, the two graphs of $y = \sin x$ and $y = \cos x$ are increasing when the angle x is located in quadrant IV.



Ans (D)

TRIGONOMETRY