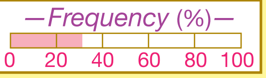
Lesson 23

PLANE FIGURES

Areas

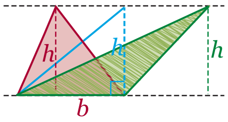


Most Valuable Points

MVP ① Area

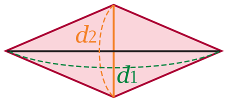
(1) Triangle ⇒ $A = \frac{1}{2} \times (\text{base}) \cdot (\text{height}) = \frac{1}{2}bh$

✻ Same base and same height ⇒ Same area



(2) Rhombus/Square (← Perpendicular diagonals)

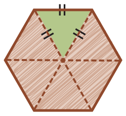
⇒ $A = \frac{1}{2} \times (\text{diagonal}) \cdot (\text{diagonal}) = \frac{1}{2}d_1d_2$

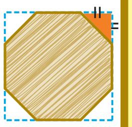


(3) Circle ⇒ $A = (\text{radius})^2 \cdot \pi = r^2\pi$

(4) Regular polygons

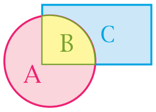
- Regular hexagon ⇒ 6 × Equilateral triangle
- Regular Octagon ⇒ Square − 4 × Isosceles right triangle





MVP ② Application

Overlapping shape ⇒
- If A + B = B + C, then A = C
- If A = C, then A + B = B + C



Most Valuable Problems

MVP 1 ★★☆

If the perimeter of a semicircular region is numerically equal to the area of the semicircular region, then what is the radius of the semicircle?

(A) 1.31 (B) 1.44 (C) 2.00

(D) 2.27 (E) 3.27

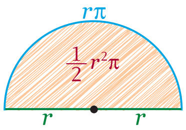
Idea! If the radius of the semicircle is r,

the perimeter of the semicircle is $2 \times r + \frac{1}{2} \times (2 \cdot r \cdot \pi) = 2r + r\pi = r(2 + \pi)$ ……①

and the area of the semicircle is $\frac{1}{2} \times (r^2 \cdot \pi) = \frac{1}{2}r^2\pi$ ……②

From ① and ②, $r(2 + \pi) = \frac{1}{2}r^2\pi \quad \therefore 2 + \pi = \frac{1}{2}r\pi$

$\therefore r\pi = 4 + 2\pi \quad \therefore r = \frac{4 + 2\pi}{\pi} = 2 + \frac{4}{\pi} = 3.2732\cdots\cdots \approx 3.27$

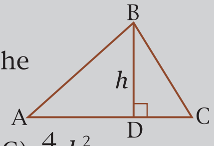


Ans (E)

MVP 2 ★☆☆

In triangle ABC, $\overline{BD} \perp \overline{AC}$ and BD = h.

If BD is 37.5% less than AC, then what is the area of triangle ABC?



(A) $\frac{2h^2}{3}$ (B) $\frac{4}{3}h^2$ (C) $\frac{4}{5}h^2$

(D) $\frac{8}{3}h^2$ (E) $\frac{5}{16}h^2$

Ans (C)

MVP 3 ★★☆

In a greater circle with center O, a semicircle is constructed using segment $\overline{AB}$ as its diameter as shown. What is the ratio of the areas of two shaded regions (A) to (B)?

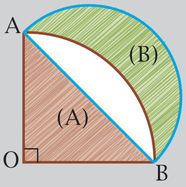
(A) $\frac{2}{\pi}$

(B) $\frac{3}{\pi}$

(C) $\frac{\pi}{3}$

(D) $\frac{\sqrt{2}}{\pi}$

(E) 1



MVP 4 ★★☆

In the figure, two medians $\overline{AM}$ and $\overline{BN}$ of triangle ABC are perpendicular. If AM = 7 and BN = 9, then what is the area of triangle ABC?

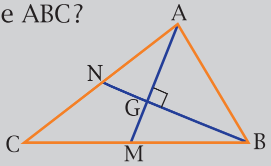
(A) 18

(B) 35

(C) 42

(D) 56

(E) 63



MVP 5 ★★☆

If the sides of an octagon ABCDEFGH alternate in length as shown, what is the area of the octagon?

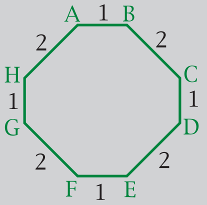
(A) $4 + 2\sqrt{2}$

(B) $5 + 4\sqrt{2}$

(C) $12 - 2\sqrt{2}$

(D) $8\sqrt{2}$

(E) $12\sqrt{2}$



MVP 6 ★★☆

In the square, the inscribed rectangle has side length 6. The length of that side is 75% of the side length of the circumscribed square. What is the sum of areas of the shaded regions?

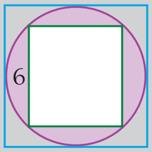
(A) 13.92

(B) 15.26

(C) 18.52

(D) 22.96

(E) 30.54



MVP 7 ★★★

In a regular hexagon ABCDEF with each side 4, P and Q are the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively. What is the area of the triangle FPQ?

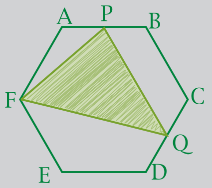
(A) 16

(B) $9\sqrt{3}$

(C) 18

(D) $10\sqrt{2}$

(E) 20



MVP 8 ★★★

An equilateral triangle with each side of 4 and a circle share a common center. The total area of the regions that are inside the circle and outside the equilateral triangle is equal to the total area of the regions that are outside the circle and inside the triangle. What is the radius of the circle?

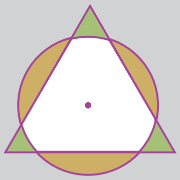
(A) 1.32

(B) 1.49

(C) 1.68

(D) 1.92

(E) 2.09



Lesson 23

Lesson 36

LOGARITHMS

Operations

—Frequency (%)—

0 20 40 60 80 100

Most Valuable Points

MVP ❶ Operations of the logarithms in same bases

(1) Addition

$$\underline{\log a + \log b} \Leftrightarrow \underline{\log ab}$$

Expanded form Single log form

ex. $\log(a+b) + \log(a-b)$

$\Rightarrow \log(a+b)(a-b)$

$= \log(a^2 - b^2)$

(2) Subtraction

$$\underline{\log a - \log b} \Leftrightarrow \underline{\log \frac{a}{b}}$$

Expanded form Single log form

ex. $\log \sin x - \log \cos x$

$\Rightarrow \log \frac{\sin x}{\cos x} = \log \tan x$

(3) Power of antilog

$$\underline{n\log a} \Leftrightarrow \underline{\log a^n}$$

Expanded form Single log form

ex. $\frac{1}{2}\log 8 \Rightarrow \log 8^{\frac{1}{2}}$

$= \log\sqrt{8} = \log 2\sqrt{2}$

MVP ❷ Change of bases.

$$\log_a b = \frac{\log b}{\log a} = \frac{\ln b}{\ln a} = \frac{1}{\log_b a} = \log_{a^n} b^n$$

Base a Base 10 Base e Base b Base a^n

ex. $\log_5 3 \Rightarrow \frac{\log 3}{\log 5} = \frac{\ln 3}{\ln 5} = \frac{1}{\log_3 5}$

$\ln 10 \Rightarrow \log_e 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$

Most Valuable Problems

MVP 1 ★★☆

If $10^x = a$ and $100^y = b$, then which of the following expressions is equivalent to $\frac{1}{2}x - y$?

(A) $\log \frac{a}{b}$

(B) $\frac{\log a}{\log b}$

(C) $\log\sqrt{\frac{a}{b}}$

(D) $\frac{\log(a-b)}{2}$

(E) $\log \frac{1}{\sqrt{a-b}}$

Idea! i) $10^x = a \Rightarrow x = \log_{10} a = \log a$

ii) $100^y = b \Rightarrow y = \log_{100} b = \log_{10^2} b = \frac{1}{2}\log_{10} b = \frac{1}{2}\log b$

From i) and ii), $\frac{1}{2}x - y = \frac{1}{2}\log a - \frac{1}{2}\log b = \frac{1}{2}(\log a - \log b) = \frac{1}{2}\log\frac{a}{b} = \log\sqrt{\frac{a}{b}}$ **Ans** (C)

MVP 2 ★☆☆

What is the value of $\log_2 2\sqrt{6} + \log_2 4\sqrt{3} - \log_2 6$?

(A) -3

(B) $-\frac{3}{2}$

(C) $\frac{1}{2}$

(D) $\frac{3}{2}$

(E) $\frac{5}{2}$

Ans (E)

MVP 3 ★☆☆

Which of the following must be true?

I. $(\log_2 4)^2 = \log_2 4^2$

II. $2^{\log\sqrt{3}} = 3^{\log\sqrt{2}}$

III. $(\log 5)^2 - (\log 2)^2 = \log\frac{5}{2}$

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II and III

MVP 4 ★☆☆

If $\log a = 0.5733$, then what is the value of $\log(0.0001 \times a^3)$?

(A) –0.2801

(B) –1.2801

(C) –2.2801

(D) –2.7199

(E) –3.7199

MVP 5 ★★☆

If $\log 2 = a$ and $\log 3 = b$, then what is the value of $\log\cos 330°$?

(A) $a + \frac{1}{2}b$

(B) $\frac{1}{2}b - a$

(C) $\frac{\sqrt{b}}{a}$

(D) $a - \frac{1}{2}b$

(E) $\frac{\sqrt{a}}{b}$

MVP 6 ★★☆

Which of the following is equivalent to the logarithmic expression $\log_2 xy^2 + 3\log_2 x - 2\log_2 y + 5\log_2\frac{1}{x^2}$?

(A) $\frac{2}{5}\log_2 x$

(B) $2\log_2 y$

(C) $3\log_2 y$

(D) $-6\log_2 x$

(E) $\frac{1}{6}\log_2 x$

MVP 7 ★☆☆

If $f(x) = \log\sqrt{x}$ and $g(x) = \frac{x}{100}$, then $f(g(x))$ could be expressed as

(A) $\log x - 1$

(B) $\log x - 2$

(C) $\frac{1}{2}\log x - 1$

(D) $\frac{1}{2}(\log x - 10)$

(E) $\frac{\log x}{50}$

MVP 8 ★★☆

What is the value of $\log_2 5 \cdot \log_{25}\frac{1}{3} \cdot \log_{\sqrt{3}} 8$?

(A) –3

(B) 6

(C) 12

(D) $-\frac{1}{3}$

(E) $-\frac{3}{4}$

Lesson
69

PARAMETRIC EQUATIONS

Graphs

—Frequency (%)—
0 20 40 60 80 100

My Very Points

MVP ❶ Domains / Ranges of the parametric functions

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ ➡ $\begin{cases} \text{The range of } f(t) \Rightarrow \text{The interval of } x & [\leftarrow \text{Domain}] \\ \text{The range of } g(x) \Rightarrow \text{The interval of } y & [\leftarrow \text{Range}] \end{cases}$

⇒ The interval of values of the eliminating parametric expression

…… Leaving the interval for x or y

ex.

① $x = 2t + 1 \Rightarrow t = \frac{x-1}{2} \rightarrow$ (All real values of t) ⇒ All values of x

② $x = t^2 - 1 \Rightarrow t^2 = x + 1 \rightarrow (t^2 \geq 0 \ \therefore x + 1 \geq 0) \Rightarrow x \geq -1$

③ $x = \sqrt{2t} + 1 \Rightarrow \sqrt{2t} = x - 1 \rightarrow (\sqrt{2t} \geq 0 \ \therefore x - 1 \geq 0) \Rightarrow x \geq 1$

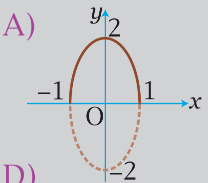
④ $x = \sin^2\theta - 2 \Rightarrow \sin^2\theta = x + 2 \rightarrow (0 \leq \sin^2\theta \leq 1,\ 0 \leq x + 2 \leq 1) \Rightarrow -2 \leq x \leq -1$

Most Valuable Problems

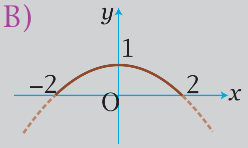
MVP 1 ★★★

What is the graph of the curve whose parametric equations are $x = 2\sin t$ and $y = \cos^2 t$?

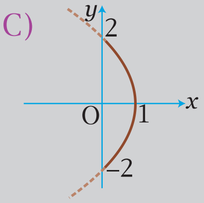
(A)



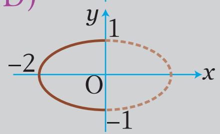
(B)



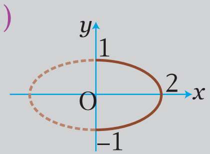
(C)



(D)



(E)



Idea! $\begin{cases} x = 2\sin t \Rightarrow \sin t = \frac{x}{2} \ \cdots\cdots ① \\ y = \cos^2 t \ \cdots\cdots ② \end{cases}$ Since $-1 \leq \sin t \leq 1$, $-1 \leq \frac{x}{2} \leq 1$ $\therefore -2 \leq x \leq 2$

From $①^2 + ②$, $\sin^2 t + \cos^2 t = \left(\frac{x}{2}\right)^2 + y$ $\therefore 1 = \frac{x^2}{4} + y$ $\therefore y = -\frac{x^2}{4} + 1\ (-2 \leq x \leq 2)$

∴ The graph of $y = -\frac{x^2}{4} + 1\ (-2 \leq x \leq 2)$ is represented by (B).

Ans (B)

MVP 2 ★★☆

Which of the following parametric equations have the same graph?

I. $\begin{cases} x = 2\sin\theta \\ y = 2\cos\theta \end{cases}$ II. $\begin{cases} x = t \\ y = \sqrt{4 - t^2} \end{cases}$ III. $\begin{cases} x = \sqrt{s} \\ y = \sqrt{4 - s} \end{cases}$

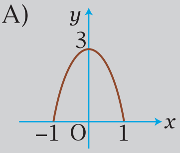
(A) I and II only (B) I and III only (C) II and III only
(D) I, II and III (E) None

Ans (E)

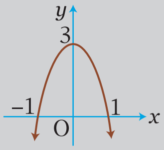
MVP 3 ★★☆

Which of the following is the graph of the parametric equations $\begin{cases} x = t^2 \\ y = 2t^2 + 3? \end{cases}$

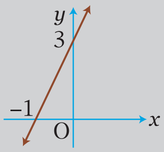
(A)



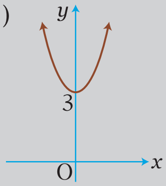
(B)



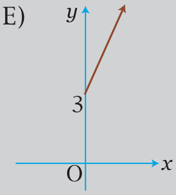
(C)



(D)



(E)



MVP 4 ★★☆

What is the domain of the function defined by the parametric equations

$$\begin{cases} x = t^2 + 2t \\ y = t^2 - 4t\,? \end{cases}$$

(A) $\{x |\ x \geq 1\}$

(B) $\{x |\ x \geq -1\}$

(C) $\{x |\ x \geq 2\}$

(D) $\{x |\ x \geq -2\}$

(E) All real numbers of x.

MVP 5 ★★☆

Which of the following is the graph of parametric equations $x = \sqrt{t+1}$ and $y = 2\sqrt{t-1}$?

(A) A straight line

(B) A portion of a straight line

(C) A parabola

(D) A portion of a parabola

(E) A portion of an ellipse.

MVP 6 ★★☆

If $x = 3\cos^2\theta$ and $y = 4\sin^2\theta$, then what is the length of the locus of points (x, y)?

(A) 5

(B) 7π

(C) 10

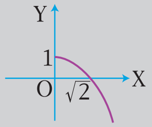
(D) 14

(E) 25π

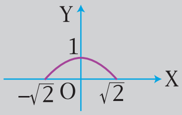
MVP 7 ★★★

If (x, y) represents a point on the graph $y = -x + 2$, then which of the following could be a portion of the graph of the set of points $\left(\frac{y}{2}, \sqrt{x}\right)$?

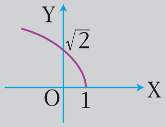
(A)



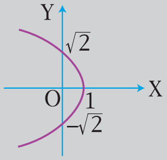
(B)



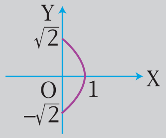
(C)



(D)



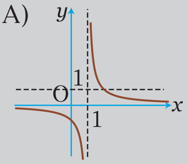
(E)



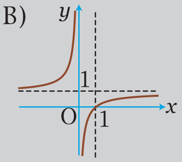
MVP 8 ★★☆

If $x = 1 + 2^t$ and $y = 1 - 2^{-t}$, which of the following is the graph of $y = f(x)$?

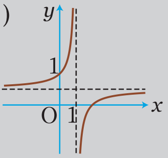
(A)



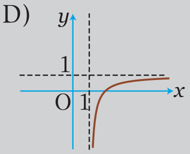
(B)



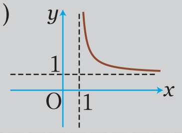
(C)



(D)



(E)



Lesson 77

POLAR EQUATIONS

Polar coordinates

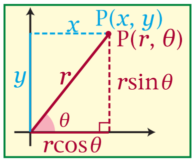
—Frequency (%)—
0 20 40 60 80 100

Most Valuable Points

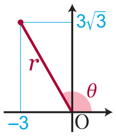
MVP ❶ Polar coordinates

Rectangular coordinate $P(x, y)$ ⇔ Polar coordinate $P(r, \theta)$

To the polar form	$x \Rightarrow r\cos\theta$ $y \Rightarrow r\sin\theta$	⇔	To the rectangular form	$r \Rightarrow \sqrt{x^2+y^2}$ $\theta \Rightarrow \left[\tan\theta = \frac{y}{x}\right]$



ex. What is the polar coordinates of the rectangular coordinates $(-3, 3\sqrt{3})$?



i) $r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$

ii) $\tan\theta = \frac{3\sqrt{3}}{-3} = -\sqrt{3}$ $\quad \therefore \theta = 120^\circ\left(= \frac{2\pi}{3}\right)$

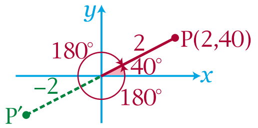
$\Rightarrow \left(6, \frac{2\pi}{3}\right)$

∗ Polar coordinates $\left(6, \frac{2\pi}{3}\right) \Rightarrow \left(6\cos\frac{2\pi}{3}, 6\sin\frac{2\pi}{3}\right) = \left(6\cdot-\frac{1}{2}, 6\cdot\frac{\sqrt{3}}{2}\right) = (-3, 3\sqrt{3})$

∗ Same point of the polar form

$P(r, \theta) = (r, \theta \pm 2n\pi) = (-r, \theta \pm (2n-1)\pi)$ (where $n = 0, 1, 2, 3, \cdots\cdots$)

ex.



$P(2, 40^\circ) = P(2, 400)^\circ = P(2, -320^\circ)$

$+360^\circ \qquad -360^\circ$

$= P(-2, 220^\circ) = P(-2, -140^\circ)$

$- \quad +180^\circ \qquad - \quad -180^\circ$

MVP ❷ Polar equations

i) $r = \sqrt{x^2+y^2} \Rightarrow r^2 = x^2 + y^2$ $\qquad$ ii) $r\cos\theta = x, \ r\sin\theta = y$

ex.

Polar equation		Rectangular Equation
$r = 5$	$r^2 = 25 \quad \therefore x^2 + y^2 = 5^2$	$x^2 + y^2 = 25$
$r = \frac{2}{\cos\theta}$	$r\cos\theta = 2 \quad \therefore x = 2$	$x = 2$
$r = 4\cos\theta$	$r^2 = 4r\cos\theta \quad \therefore x^2 + y^2 = 4x$	$x^2 + y^2 - 4x = 0$

Most Valuable Problems

MVP 1 ★★☆

What is the polar form of the curve $xy = 2$?

(A) $r^2\sin2\theta = 1$ (B) $r^2\sin2\theta = 4$ (C) $r^2\sin2\theta = 8$

(D) $r^2\sin2\theta = \frac{1}{4}$ (E) $r^2\sin2\theta = \frac{1}{16}$

Idea! From the curve $xy = 2$, $(r\cos\theta)(r\sin\theta) = 2$ ($\leftarrow x = r\cos\theta$ and $y = r\sin\theta$)

$\therefore r^2\sin\theta\cdot\cos\theta = 2 \quad \therefore r^2(2\sin\theta\cdot\cos\theta) = 4 \quad \therefore r^2\sin2\theta = 4$

Ans (B)

MVP 2 ★☆☆

What are the polar coordinates of the point $(1, -\sqrt{3})$?

(A) $\left(2, \frac{5\pi}{3}\right)$ (B) $\left(2, \frac{11\pi}{6}\right)$ (C) $\left(\sqrt{2}, \frac{2}{3}\pi\right)$

(D) $\left(\sqrt{2}, \frac{5\pi}{3}\right)$ (E) $\left(2\sqrt{2}, \frac{11}{6}\pi\right)$

Ans (A)

MVP 3 ★★☆

Which of the following points of polar coordinates is different from the others?

(A) (–2, –150°)
(B) (2, –330°)
(C) (2, 30°)
(D) (–2, 210°)
(E) (–2, 390°)

MVP 4 ★★☆

A and B are two points on the circle O. In polar coordinates, A is the point $(6, \frac{\pi}{4})$ and B is the point $(6, \frac{5\pi}{6})$. What is the length of arc $\overset{\frown}{AB}$ nearest to the integer?

(A) 10
(B) 11
(C) 12
(D) 13
(E) 14

MVP 5 ★★★

Which of the following equations in rectangular form is equivalent to $r^2 = 36\sec 2\theta$?

(A) $x - y = \pm 6$
(B) $\frac{x^2}{6^2} - \frac{y^2}{6^2} = 1$
(C) $y = \pm\frac{6}{x}$
(D) $x = \pm 6$
(E) $x^2 + y^2 = 6^2$

MVP 6 ★★☆

How many intersection points are in the two graphs of $r = \cos\theta$ and $r = \sin 2\theta$?

(A) None
(B) One
(C) Two
(D) Three
(E) Four

MVP 7 ★★☆

What is the distance of two points of intersection of two graphs $r = 2$ and $r = \frac{1}{\sin\theta}$?

(A) 1
(B) $\frac{\pi}{3}$
(C) $\sqrt{3}$
(D) 2
(E) $2\sqrt{3}$

MVP 8 ★★★

What is the length of the line segment whose endpoints have polar coordinates (–3, 75°) and (4, –25°)?

(A) 4.12
(B) 4.56
(C) 4.97
(D) 5.29
(E) 5.88

Lesson 85

SPACE

Spheres

—Frequency (%)—
0 20 40 60 80 100

Most Valuable Points

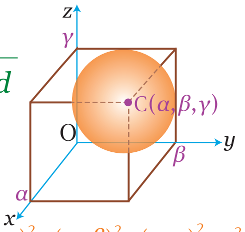
MVP ❶ Equations of the spheres

(1) General form ······ $x^2+y^2+z^2+ax+by+cz+d=0$

$$\Rightarrow \begin{cases} ① \text{ Center} = C\left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right) \\ ② \text{ Radius} = \sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2-d} \end{cases}$$

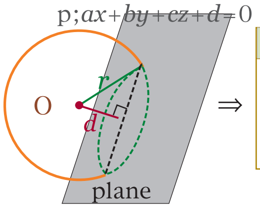
(2) Standard form ······ $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$

$$\Rightarrow \begin{cases} ① \text{ Center} = C(\alpha, \beta, \gamma) \\ ② \text{ Radius} = r \end{cases}$$



$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$

MVP ❷ Relationships of a circle and a line



⇒

Distance of the center of a circle and a line ⪌ Radius

$$\underset{|O\sim P|}{d} = \frac{|a\alpha+b\beta+c\gamma+d|}{\sqrt{a^2+b^2+c^2}} \begin{cases} < r \Rightarrow \text{Intersecting } (\leftarrow \text{Circle}) \\ = r \Rightarrow \text{Tangent} \\ > r \Rightarrow \text{No intersecting} \end{cases}$$

Most Valuable Problems

MVP 1 ★★☆

If a cube is inscribed in the sphere $x^2+y^2+z^2+2x-4z+2=0$, then what is the volume of the cube?

(A) 1 (B) $2\sqrt{2}$ (C) 8
(D) $3\sqrt{3}$ (E) 27

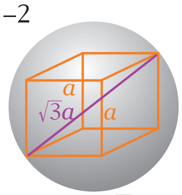
Idea! From the sphere $x^2+y^2+z^2+2x-4z+2=0$, $x^2+2x+y^2+z^2-4z=-2$

$\therefore (x+1)^2+y^2+(z-2)^2=1+4-2$ $\therefore (x+1)^2+y^2+(z-2)^2=(\sqrt{3})^3$

$\therefore$ The radius of the sphere is $\sqrt{3}$.

If each edge of the inscribed cube is a,

the length of the space diagonal is $\sqrt{a^2+a^2+a^2}=\sqrt{3}a$



Since the diameter of the sphere is equal to the space diagonal of the cube, $2\sqrt{3}=\sqrt{3}a$

$\therefore a=2$ $\therefore$ The volume of the cube is $a^3=2^3=8$

Ans (C)

MVP 2 ★☆☆

A sphere $(x+1)^2+(y-2)^2+(z-4)^2=4$ is tangent to

(A) the y-axis (B) the z-axis (C) the xy-line
(D) the yz-plane (E) the xz-plane

Ans (E)

MVP 3 ★★☆

What is the equation of the set of points that are 4 units from point (–1, 2, –3)?

(A) $-x + \frac{y}{2} - \frac{z}{3} = 4$

(B) $(-x)^2 + (2y)^2 + (-3z)^2 = 4^2$

(C) $(x-1)^2 + (y+2)^2 + (z-3)^2 = 2^2$

(D) $x^2 + y^2 + z^2 + 2x - 4y + 6z = 4^2$

(E) $x^2 + y^2 + z^2 + 2x - 4y + 6z - 2 = 0$

MVP 4 ★★☆

Which of the following points lies inside the sphere, with center at the origin, that passes through the point (–1, 2, 3)?

(A) (0, 0, 5)

(B) (0, 2, 4)

(C) (2, –2, 2)

(D) $(3, \sqrt{5}, 0)$

(E) $(-1, 2, \sqrt{10})$

MVP 5 ★★☆

What is the volume of the sphere of $x^2 + y^2 + z^2 - 2x + 6y - 8z + 1 = 0$?

(A) 26.2

(B) 46.8

(C) 78.5

(D) 104.7

(E) 523.6

MVP 6 ★★☆

A sphere has points A(–3, 4, 2) and B(1, 2, –2) as endpoints of a diameter. Which of the following is the equation of the sphere?

(A) $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$

(B) $x^2 + y^2 + z^2 - 2x + 6y + 1 = 0$

(C) $x^2 + y^2 + z^2 + 2x - 6y + 7 = 0$

(D) $x^2 + y^2 + z^2 - x + 3y + 7 = 0$

(E) $x^2 + y^2 + z^2 + x - 3y + 7 = 0$

MVP 7 ★★☆

What is the distance of two intersection points of a sphere $(x-2)^2 + (y+1)^2 + (z+3)^2 = 3^2$ and the z-axis?

(A) 2

(B) 4

(C) 6

(D) 8

(E) 10

MVP 8 ★★★

A plane $2x + y + 3z = 9$ intersects a sphere $x^2 + y^2 + z^2 - 2x - 2y = 7$ in space and produces a circle. What is the area of the circle?

(A) 3.14

(B) 6.28

(C) 14.05

(D) 15.71

(E) 21.06

ii) Obtuse triangle ⇒

① $7 > k;\ 7^2 > 4^2 + k^2$ ∴ $49 > 16 + k^2$

∴ $k^2 > 33$ ∴ $k = 5, 4, 3, 2, \cdots\cdots$

② $k > 7;\ k^2 > 7^2 + 4^2$ ∴ $k^2 > 49 + 16$

∴ $k^2 > 65$ ∴ $k = 9, 10, 11, \cdots\cdots$

From i) and ii), $k = 4, 5, 9, 10$

∴ The number of integer values of k is 4.

Ans (A)

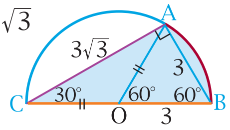
4 In the figure, $\overline{BC}$ is a diameter. ∴ $\angle A = 90^\circ$

In 30°–60°–90° right triangle ABC,

$CB = 2 \cdot 3 = 6$ and $AC = 3\sqrt{3}$

∴ The perimeter of the shaded region is

$\overset{\frown}{AB} + BC + CA$

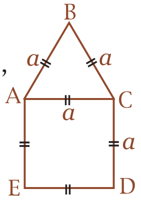


$$= \frac{60^\circ}{360^\circ} \times (2 \cdot 3 \cdot \pi) + 2 \cdot 3 + 3\sqrt{3}$$

$$= \pi + 6 + 3\sqrt{3} = 14.337745\cdots\cdots \approx 14.34$$

Ans (D)

5 In the figure, all sides of the pentagon ABCDE are the same lengths.



If each side of the pentagon is a, the area is $a^2 + \frac{\sqrt{3}}{4}a^2$.

$$∴ \left(1 + \frac{\sqrt{3}}{4}\right)a^2 = 100$$

$$∴ a^2 = \frac{100}{1 + \frac{\sqrt{3}}{4}} \quad ∴ a = \sqrt{\frac{100}{1 + \frac{\sqrt{3}}{4}}} = 8.3536\cdots$$

∴ The perimeter of the pentagon is

$5a = 5 \times 8.3536\cdots = 41.768\cdots \approx 41.8$

Ans (E)

6 If the length of the fourth side is x, since $1 < 2 < 5$,

i) The longest side is 5

⇒ $x + 1 + 2 > 5$ ∴ $x > 2$

ii) The longest side is x

⇒ $1 + 2 + 5 > x$ ∴ $x < 8$

From i) and ii), $2 < x < 8$

∴ The interval of the perimeter is

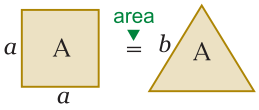
$(1 + 2 + 5) + 2 < P < (1 + 2 + 5) + 8$

∴ $10 < P < 16$

∴ The perimeter of the quadrilateral could be 11.

Ans (B)

7 In the figure, if each side of the square and the equilateral triangle are a and b respectively,



$$\begin{cases} A = a^2 & ∴ a = \sqrt{A} \\ A = \frac{\sqrt{3}}{4}b^2 & ∴ b^2 = \frac{4A}{\sqrt{3}} \quad ∴ b = \frac{2\sqrt{A}}{\sqrt[4]{3}} \end{cases}$$

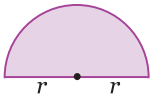
∴ The ratio of perimeters of the square to the equilateral triangle is $4a : 3b$

$$= 4\sqrt{A} : 3 \cdot \frac{2\sqrt{A}}{\sqrt[4]{3}} = 4 : \frac{6}{\sqrt[4]{3}} = 2 : \frac{3}{\sqrt[4]{3}}$$

$$= 2 \cdot \sqrt[4]{3} : 3 = 0.877\cdots\cdots \approx 0.88$$

Ans (D)

8 If the radius of the semicircle is r, the area is $\frac{1}{2} \times r^2\pi = 10$ ∴ $r^2\pi = 20$



$$∴ r^2 = \frac{20}{\pi} \quad ∴ r = \sqrt{\frac{20}{\pi}} = \frac{2\sqrt{5}}{\sqrt{\pi}}$$

∴ The perimeter of the semicircular region is $2r + \frac{1}{2} \times 2r\pi = 2r + r\pi = (2 + \pi)r$

$$= (2 + \pi) \cdot \frac{2\sqrt{5}}{\sqrt{\pi}}$$

If each side of the square is a,

$$4a = (2 + \pi) \cdot \frac{2\sqrt{5}}{\sqrt{\pi}}$$

$$∴ a = \frac{2\sqrt{5}(2 + \pi)}{4\sqrt{\pi}} = \frac{\sqrt{5}(2 + \pi)}{2\sqrt{\pi}}$$

∴ The area of the square is $a^2 = \frac{5(2 + \pi)^2}{4\pi}$

$= 10.51854\cdots\cdots \approx 10.52$

Ans (B)

Lesson 23 PLANE FIGURES

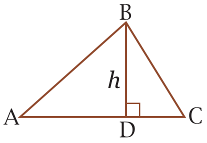
Areas

2 Since h is 37.5 % less than AC,

$$h = \left(1 - \frac{37.5}{100}\right) \cdot AC$$

$$= \left(1 - \frac{3}{8}\right) AC = \frac{5}{8} AC$$

$$∴ AC = \frac{8}{5}h$$

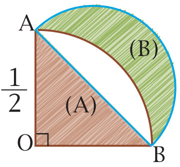


∴ The area of triangle ABC is $\frac{1}{2} \times \frac{8}{5}h \cdot h$

$$= \frac{4}{5}h^2$$

Ans (C)

3 In the figure, if OA = OB = 1,
AB = $\sqrt{1^2+1^2}=\sqrt{2}$



∴ The area of (A) is $\frac{1}{2}\cdot 1^2=\frac{1}{2}$

and the area of (B) is

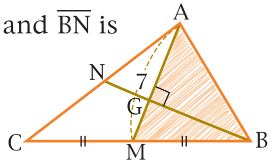
$$\frac{1}{2}\cdot 1^2+\frac{1}{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^2\pi-\frac{1}{4}\cdot 1^2\pi$$

$$=\frac{1}{2}+\frac{\pi}{4}-\frac{\pi}{4}=\frac{1}{2}$$

∴ (A) : (B) = $\frac{1}{2}:\frac{1}{2}$ = 1

Ans (E)

4 In triangle ABC, the intersection point G of the two medians $\overline{AM}$ and $\overline{BN}$ is the center of gravity.



∴ NG : GB = 1 : 2

∴ GB = $\frac{2}{3}\times 9=6$

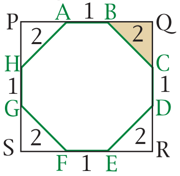
∴ The area of triangle ABM is $\frac{1}{2}\cdot 7\cdot 6=21$

Since CM = BM, the area of triangle ACM is the same as the area of triangle ABM.

∴ The area of triangle ABC is 2 × 21 = 42.

Ans (C)

5 In the figure, isosceles right triangle BQC,
BQ = QC = $\frac{2}{\sqrt{2}}=\sqrt{2}$



∴ The area of the octagon is

$$(1+2\sqrt{2})^2-4\times\frac{1}{2}(\sqrt{2})^2$$

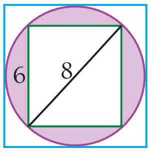
$$=(1+4\sqrt{2}+8)-2\cdot 2$$

$$=(9+4\sqrt{2})-4=5+4\sqrt{2}$$

Ans (B)

6 If the side length of the circumscribed square is a, $a\times\frac{75}{100}=6$ ∴ $a\times\frac{3}{4}=6$

∴ $a=8$ ∴ The length of a diagonal of the inscribed rectangle is $a=8$.



∴ The other side length of the rectangle is $\sqrt{8^2-6^2}$

$=\sqrt{64-36}=\sqrt{28}=2\sqrt{7}$

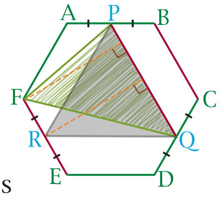
∴ The area of the shaded regions is

$4^2\pi-6\cdot 2\sqrt{7}=16\pi-12\sqrt{7}=18.5165\cdots\cdots$

≈ 18.52

Ans (C)

7 In the figure, BC = 4
and AD = 2 × 4 = 8



Since $\overline{BC}\parallel\overline{AD}$,

PQ = $\frac{1}{2}(4+8)=6$

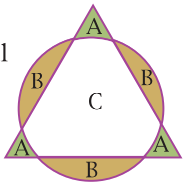
In triangle FPQ, if $\overline{PQ}$ is the base, the altitudes of △FPQ and △RPQ are the same.(← R is the midpoint of $\overline{EF}$.)

∴ The area of triangle FPQ is equal to the area of equilateral triangle RPQ with each side 6.

∴ The area of triangle FPQ = $\frac{\sqrt{3}}{4}\cdot 6^2=9\sqrt{3}$

Ans (B)

8 In the figure, 3A = 3B.



The area of the equilateral triangle is C + 3A.

The area of the circle is C + 3B.

Since 3A = 3B, the area of the equilateral triangle is equal to the area of the circle.

∴ $\frac{\sqrt{3}}{4}$. $4^2=r^2\pi$ ∴ $r^2=\frac{4\sqrt{3}}{\pi}$

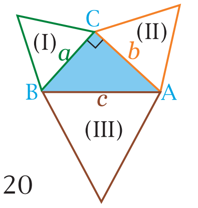
∴ $r=\sqrt{\frac{4\sqrt{3}}{\pi}}=1.48503\cdots\cdots\approx 1.49$

Ans (B)

Lesson 24 PYTHAGOREAN THEOREM

Concepts

2 In the figure, if BC = a, CA = b and AB = c, the areas of 3 equilateral triangles are



$\frac{\sqrt{3}}{4}a^2=\sqrt{3}$, $\frac{\sqrt{3}}{4}b^2=4\sqrt{3}$

and $\frac{\sqrt{3}}{4}c^2=5\sqrt{3}$

∴ $a^2=4$, $b^2=16$ and $c^2=20$

Since $a^2+b^2=c^2$, the triangle ABC is a right triangle with C = 90°.(← $a=2$, $b=4$, $c=2\sqrt{5}$)

∴ The area the of triangle ABC is

$\frac{1}{2}\times a\cdot b=\frac{1}{2}\times 2\cdot 4=4$

Ans (B)

Lesson 36 LOGARITHMS

Operations

2 $\log_2 2\sqrt{6} + \log_2 4\sqrt{3} - \log_2 6$

$$= \log_2 \frac{2\sqrt{6}\cdot 4\sqrt{3}}{6} = \log_2 \frac{8\sqrt{18}}{6}$$

$$= \log_2 \frac{8\cdot 3\sqrt{2}}{6} = \log_2 4\sqrt{2} = \log_2 2^{\frac{5}{2}} = \frac{5}{2}$$

Ans (E)

3 I. $(\log_2 4)^2 = 2^2 = 4$ and $\log_2 4^2 = \log_2 16 = 4$

$\therefore (\log_2 4)^2 = \log_2 4^2$

II. If $2^{\log\sqrt{3}} = 3^{\log\sqrt{2}}$, $\log 2^{\log\sqrt{3}} = \log 3^{\log\sqrt{2}}$

$\therefore (\log\sqrt{3})(\log 2) = (\log\sqrt{2})(\log 3)$

$\therefore \cancel{\frac{1}{2}}\log 3\cdot\log 2 = \cancel{\frac{1}{2}}\log 2\cdot\log 3$

$\therefore 2^{\log\sqrt{3}} = 3^{\log\sqrt{2}}$

III. $(\log 5)^2 - (\log 2)^2$

$= (\log 5 + \log 2)(\log 5 - \log 2)$

$= \log 10\cdot(\log 5 - \log 2) = 1\cdot(\log 5 - \log 2)$

$= \log\frac{5}{2}$

Ans (E)

4 $\log(0.0001\times a^3) = \log 0.0001 + \log a^3$

$= \log 10^{-4} + 3\log a = (-4) + 3\times 0.5733$

$= -4 + 1.7199 = -2.2801$

Ans (C)

5 $\log\cos 330^\circ = \log\cos(330^\circ - 360^\circ)$

$= \log\cos(-30^\circ) = \log\cos 30^\circ = \log\frac{\sqrt{3}}{2}$

$= \log\sqrt{3} - \log 2 = \frac{1}{2}\log 3 - \log 2 = \frac{1}{2}b - a$

Ans (B)

6 $\log_2 xy^2 + 3\log_2 x - 2\log_2 y + 5\log_2\frac{1}{x^2}$

$= \log_2 xy^2 + 3\log_2 x - 2\log_2 y + 5\log_2 x^{-2}$

$= \log_2 xy^2 + \log_2 x^3 - \log_2 y^2 + \log_2 x^{-10}$

$= \log_2\frac{x\cancel{y^2}\cdot x^3\cdot x^{-10}}{\cancel{y^2}} = \log_2 x^{-6} = -6\log_2 x$

Ans (D)

7 $f(g(x)) = \log\sqrt{\frac{x}{100}} = \log\frac{\sqrt{x}}{10} = \log\sqrt{x} - \log 10$

$= \log x^{\frac{1}{2}} - 1 = \frac{1}{2}\log x - 1$

Ans (C)

8 $\log_2 5\cdot\log_{25}\frac{1}{3}\cdot\log_{\sqrt{3}}8$

$$= \frac{\log 5}{\log 2}\cdot\frac{\log\frac{1}{3}}{\log 25}\cdot\frac{\log 8}{\log\sqrt{3}}$$

$$= \frac{\log 5}{\log 2}\cdot\frac{\log 3^{-1}}{\log 5^2}\cdot\frac{\log 2^3}{\log 3^{\frac{1}{2}}}$$

$$= \frac{\cancel{\log 5}}{\cancel{\log 2}}\cdot\frac{-\cancel{\log 3}}{2\cancel{\log 5}}\cdot\frac{3\cancel{\log 2}}{\frac{1}{2}\cancel{\log 3}} = \frac{-3}{2\cdot\frac{1}{2}} = -3$$

Ans (A)

Lesson 37 LOGARITHMS

Functions / Equations

2 From the equation $\ln(x+2) = 1 + \ln x$,

Anti log > 0 $\Rightarrow \begin{cases} x+2>0 \Rightarrow x>-2 \cdots\cdots ① \\ x>0 \cdots\cdots ② \end{cases}$

From ① and ②, $x > 0$

And $\ln(x+2) = \ln e + \ln x$

$\therefore \ln(x+2) = \ln ex \quad \therefore x+2 = ex$

$\therefore (e-1)x = 2 \quad \therefore x = \frac{2}{e-1}$

Since $\frac{2}{e-1} > 0,\ x = \frac{2}{e-1}$

Ans (E)

3 From the equation $\log_3 x = \log_x 5$,

$\frac{\ln x}{\ln 3} = \frac{\ln 5}{\ln x} \quad \therefore (\ln x)^2 = \ln 3\cdot\ln 5$

$\therefore \ln x = \pm\sqrt{\ln 3\cdot\ln 5} = \pm 1.32972\cdots \approx \pm 1.33$

$\therefore x \approx e^{1.33}, e^{-1.33} \approx 3.781, 0.264$

$\therefore$ The sum of roots of the equation is

$3.781 + 0.264 = 4.045$

Ans (A)

4 $f(x) = \frac{100^{\log x}}{\log 100^x} = \frac{100^{\log_{100}x^2}}{x\log 100}$

$= \frac{x^2}{x\cdot 2} = \frac{x}{2}$ (← Antilog $x > 0$)

$\therefore f(\frac{1}{5}) = \frac{\frac{1}{5}}{2} = \frac{1}{10}$

$\therefore \log f((\frac{1}{5})) = \log\frac{1}{10} = \log 10^{-1} = -1$

Ans (B)

5 From the equation $3^{x+3} = 135$,

$\log 3^{x+3} = \log 135$

$\therefore$ The graph of $(x-1)^2 + \dfrac{y^2}{2^2} = 1$

[← an ellipse with center at (1, 0)]

is represented by (A).

Ans (A)

8 From $(x-y, x+y)$,

if $\begin{cases} X = x-y \cdots\cdots ① \\ Y = x+y \cdots\cdots ②, \end{cases}$

From ① + ②, $X + Y = 2x \quad \therefore x = \dfrac{X+Y}{2}$

From ① − ②, $X - Y = -2y \quad \therefore y = \dfrac{Y-X}{2}$

Since $x^2 + y^2 = 1$, $\left(\dfrac{X+Y}{2}\right)^2 + \left(\dfrac{Y-X}{2}\right)^2 = 1$

$\therefore \dfrac{X^2+2XY+Y^2}{4} + \dfrac{Y^2-2XY+X^2}{4} = 1$

$\therefore \dfrac{2(X^2+Y^2)}{4} = 1 \quad \therefore X^2 + Y^2 = 2$

$\therefore x^2 + y^2 = 2$ represents the set of all points (x, y).

Ans (A)

Lesson 69 PARAMETRIC EQUATIONS

Graphs

2 I. $\begin{cases} x = 2\sin\theta \Rightarrow x^2 = 4\sin^2\theta \cdots\cdots ① \\ y = 2\cos\theta \Rightarrow y^2 = 4\cos^2\theta \cdots\cdots ② \end{cases}$

From ① + ②, $x^2 + y^2 = 4(\sin^2\theta + \cos^2\theta) = 4$

$\therefore x^2 + y^2 = 2^2$ (← Circle)

II. $\begin{cases} x = t \quad \Rightarrow t = x \cdots\cdots ① \\ y = \sqrt{4-t^2} \Rightarrow y = \sqrt{4-t^2} \cdots\cdots ② \end{cases}$

Since $\sqrt{4-t^2} \geq 0$, $y \geq 0$

From ① → ②, $y = \sqrt{4-x^2} \quad \therefore y^2 = 4 - x^2$

$\therefore x^2 + y^2 = 2^2$ (← $y \geq 0$; Semicircle)

III. $\begin{cases} x = \sqrt{s} \Rightarrow s = x^2 \cdots\cdots ① \quad \text{Since } \sqrt{s} \geq 0,\ x \geq 0 \\ y = \sqrt{4-s} \Rightarrow y = \sqrt{4-s} \cdots\cdots ② \end{cases}$

Since $\sqrt{4-s} \geq 0$, $y \geq 0$

From ① → ②, $y = \sqrt{4-x^2} \quad \therefore y^2 = 4 - x^2$

$\therefore x^2 + y^2 = 2^2$ (← $x \geq 0$, $y \geq 0$; Quadrant)

$\therefore$ All the equations have different graphs.

Ans (E)

3 $\begin{cases} x = t^2 \cdots\cdots ① \Rightarrow \text{Since } t^2 \geq 0,\ x \geq 0 \\ y = 2t^2 + 3 \cdots\cdots ②, \end{cases}$

From ① → ②, $y = 2x + 3$ $(x \geq 0)$

$\therefore$ The graph of $y = 2x + 3$ $(x \geq 0)$ is represented by (E).

Ans (E)

4 From $\begin{cases} x = t^2 + 2t \\ y = t^2 - 4t, \end{cases}$

$x = t^2 + 2t = (t+1)^2 - 1 \geq -1 \quad \therefore x \geq -1$

$\therefore$ The domain of the function is $\{x \mid x \geq -1\}$

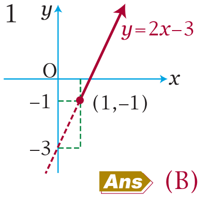
Ans (B)

5 $\begin{cases} x = \sqrt{t} + 1 \Rightarrow \sqrt{t} = x - 1 \geq 0 \cdots\cdots ① \\ \qquad \text{Since } \sqrt{t} \geq 0,\ x \geq 1 \\ y = 2\sqrt{t} - 1 \Rightarrow 2\sqrt{t} = y + 1 \geq 0 \cdots\cdots ② \\ \qquad \text{Since } 2\sqrt{t} \geq 0,\ y \geq -1 \end{cases}$

From ① → ②, $2(x-1) = y + 1$

$\therefore y = 2x - 3$ $(x \geq 1)$

The graph is a portion of a straight line.

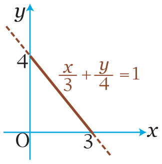


Ans (B)

6 $\begin{cases} x = 3\cos^2\theta \Rightarrow \cos^2\theta = \dfrac{x}{3} \cdots\cdots ① \\ \qquad \text{Since } 0 \leq \cos^2\theta \leq 1,\ 0 \leq x \leq 3 \\ y = 4\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{y}{4} \cdots\cdots ② \\ \qquad \text{Since } 0 \leq \sin^2\theta \leq 1,\ 0 \leq y \leq 4 \end{cases}$

From ① + ②, $\cos^2\theta + \sin^2\theta = \dfrac{x}{3} + \dfrac{y}{4}$

$\therefore \dfrac{x}{3} + \dfrac{y}{4} = 1$ $(0 \leq x \leq 3,\ 0 \leq y \leq 4)$



In the graph, the length of the locus of the points is $\sqrt{3^2 + 4^2} = 5$

Ans (A)

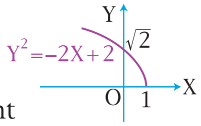
7 From $\left(\dfrac{y}{2}, \sqrt{x}\right)$, if $X = \dfrac{y}{2}$ and $Y = \sqrt{x}$,

since $\sqrt{x} \geq 0$, $Y \geq 0 \quad \therefore y = 2X$ and $x = Y^2$

From $y = -x + 2$, $2X = -Y^2 + 2$

$\therefore Y^2 = -2X + 2 \quad \therefore Y^2 = 4 \cdot -\dfrac{1}{2}(X - 1)$ (←$Y \geq 0$)

$\therefore$ The set of points $\left(\dfrac{y}{2}, \sqrt{x}\right)$ represents a portion of parabola with vertex point (1, 0) as shown.



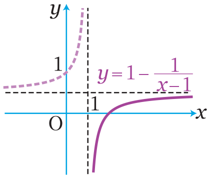
Ans (C)

8
$$\begin{cases} x = 1 + 2^t \Rightarrow 2^t = x - 1 \cdots\cdots ① \\ \qquad\qquad \text{Since } 2^t > 0 \quad \therefore x > 1 \\ y = 1 - 2^{-t} \Rightarrow y = 1 - \dfrac{1}{2^t} \cdots\cdots ② \end{cases}$$

From ① → ②,

$y = 1 - \dfrac{1}{x-1}$ $(x > 1)$

∴ The graph of $y = f(x)$ is as shown.



Ans (C)

Lesson 70 SETS
Concept

2 From A = {1, 3} and B = {0, 1},

$a \in \{1, 3\}$ and $b \in \{0, 1\}$

$\therefore c = a + 2b$

$= 1 + 2 \cdot 0,\ 1 + 2 \cdot 1,\ 3 + 2 \cdot 0,\ 3 + 2 \cdot 1$

$= 1, 3, 3, 5 \qquad \therefore c = \{1, 3, 5\}$

$\therefore A \blacktriangle B = \{c \mid c = a + 2b\} = \{1, 3, 5\}$

∴ The set of A ▲ B has 3 elements.

Ans (B)

3 S = {P | P is a prime number less than 10}

= {2, 3, 5, 7}

∴ The number of subsets of S, excluding itself is $2^4 - 1 = 16 - 1 = 15$

Ans (C)

4 I. The set of the even integers

= {⋯⋯, −4, −2, 0, 2, 4, 6, ⋯⋯}

∴ There is no least element in the set.

II. The set of the prime numbers

= { 2, 3, 5, 7, 11, 13, ⋯⋯}

∴ 2 is the least element in the set.

III. The set of positive rational numbers

$= \left\{\cdots\cdots, \dfrac{1}{1000}, \dfrac{1}{100}, \dfrac{1}{10}, 1, \dfrac{3}{2}, \cdots\cdots\right\}$

∴ There is no least element in the set.

Ans (B)

5 Since $x = \left(\dfrac{1}{64}\right)^{\frac{1}{n}} = \left(\dfrac{1}{2^6}\right)^{\frac{1}{n}} = \left(2^{-6}\right)^{\frac{1}{n}} = 2^{-\frac{6}{n}}$,

x is a positive integer, where n is a negative factor of 6. [← $-\dfrac{6}{n}$ is a positive integer]

$\therefore n = -1, -2, -3$ and -6

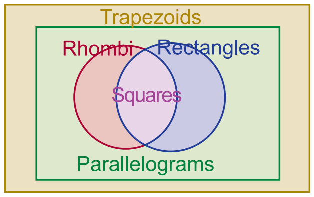
$\therefore x = 2^{-\frac{6}{(-1)}}, 2^{-\frac{6}{(-2)}}, 2^{-\frac{6}{(-3)}}$ and $2^{-\frac{6}{(-6)}}$

$= (2^6, 2^3, 2^2$ and $2 =)$ 64, 8, 4 and 2

∴ There are four positive integer elements in the set A.

Ans (C)

6



In the diagram,

(A) {Rhombi} ⊂ {Trapezoids} ∴ A ⊂ B

(B) {Parallelograms} ⊂ {Trapezoids} ∴ E ⊂ B

(C) {Squares} ⊂ {Rhombi} ∴ C ⊂ A

∴ A ∩ C = C

(D) {Rhombi} ∩ {Rectangles} = {Squares}

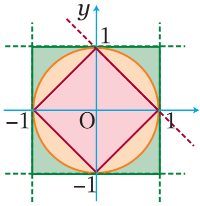
∴ A ∩ D = C

(E) {Rectangles} ⊂ {Parallelograms} ∴ D ⊂ E

∴ D ∪ E = E

Ans (E)

7 In the figure,



$A = \{(x, y) \mid x^2 + y^2 < 1\}$

⇒ Inside of the circle with diameter of 2

$B = \{(x, y) \mid |x| + |y| < 1\}$

⇒ Inside of the rhombus with diagonal of 2.

$C = \{(x, y) \mid |x| < 1$ and $|y| < 1\}$

⇒ Inside of the square with side of 2.

∴ B ⊂ A ⊂ C

Ans (B)

8 The number of subsets with three elements is ${}_7C_3 = \dfrac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$

These 35 subsets have all (35 × 3 =)105 elements.

Each of 1, 2, 3,⋯, 7 has (105 ÷ 7 =)15 times in appearance.

∴ The sum of the elements of the subsets A is 15 × (1 + 2 + ⋯ + 7) = 28 × 15 = 420

Ans (E)

(A) Center $\Rightarrow (-1, 2)$

(B) Foci $\Rightarrow (0-1, \sqrt{2^2+1^2}+2) = (-1, 2 \pm \sqrt{5})$

(C) Length of transverse axis $\Rightarrow 2 \times 2 = 4$

(D) Eccentricity $\Rightarrow \frac{\sqrt{5}}{2}$

(E) Length of latus rectum $\Rightarrow \frac{2(-1)^2}{2} = 1$

Ans (E)

POLAR EQUATIONS

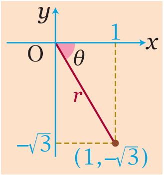
Polar coordinates

2 From the point $(1, -\sqrt{3})$,

$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-\sqrt{3})^2} = 2$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$\therefore \theta = -\frac{\pi}{3}$



Since the point lies in the quadrant IV, $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\therefore$ The polar coordinates of the point $(1, -\sqrt{3})$ is $\left(2, \frac{5\pi}{3}\right)$.

Ans (A)

3 Since $(r, \theta) = (r, \theta \pm 2n\pi) = (-r, \theta \pm (2n-1)\pi)$ where r is an integer,

$$(2, 30^\circ) = \left[\begin{matrix}(2, 30^\circ+360^\circ) = (2, 30^\circ-360^\circ) \\ (-2, 30^\circ+180^\circ) = (-2, 30^\circ-180^\circ)\end{matrix}\right.$$

$$\therefore (2, 30^\circ) = \left[\begin{matrix}(2, 390^\circ) = (2, -330^\circ) \\ (-2, 210^\circ) = (-2, -150^\circ)\end{matrix}\right.$$

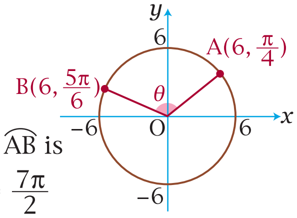
$\therefore$ The point $(-2, 390^\circ)$ is different from the others.

Ans (E)

4 In the figure,

$\theta = \frac{5\pi}{6} - \frac{\pi}{4}$

$= \frac{10\pi - 3\pi}{12} = \frac{7\pi}{12}$



$\therefore$ The length of arc $\overset{\frown}{AB}$ is

$a = r\theta = \cancel{6} \times \frac{7\pi}{\cancel{12}_{2}} = \frac{7\pi}{2}$

$= 10.9956\cdots\cdots \approx 11.$

Ans (B)

5 From the equation $r^2 = 36\sec 2\theta$,

$r^2 = 36 \cdot \frac{1}{\cos 2\theta}$

$\therefore r^2 \cdot \cos 2\theta = 36 \quad \therefore r^2(\cos^2\theta - \sin^2\theta) = 36$

$\therefore (r\cos\theta)^2 - (r\sin\theta)^2 = 36 \quad \therefore x^2 - y^2 = 36$

$\therefore \frac{x^2}{6^2} - \frac{y^2}{6^2} = 1$ (← Hyperbola)

Ans (B)

6 From $\begin{cases} r = \cos\theta \cdots\cdots ① \\ r = \sin 2\theta \cdots\cdots ②,\end{cases}$

Since ① = ②, $\cos\theta = \sin 2\theta$

$\therefore \cos\theta = 2\sin\theta \cdot \cos\theta$

$\therefore 2\sin\theta \cdot \cos\theta - \cos\theta = 0$

$\therefore \cos\theta(2\sin\theta - 1) = 0$

$\therefore \cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}$ and $\frac{5\pi}{6}$

$\therefore (r, \theta) = (\cos\theta, \theta)$

$= \left(0, \frac{\pi}{2}\right), \left(0, \frac{3\pi}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{5\pi}{6}\right)$

Since $\left(0, \frac{\pi}{2}\right) = \left(0, \frac{3\pi}{2}\right)$,

there are 3 points of intersection in two graphs of $r = \cos\theta$ and $r = \sin 2\theta$

Ans (D)

7 i) $r = 2 \Rightarrow r^2 = 4 \quad \therefore r^2(\cos^2\theta + \sin^2\theta) = 4$

$\therefore (r\cos\theta)^2 + (r\sin\theta)^2 = 4 \quad \therefore x^2 + y^2 = 4$

$\therefore x^2 + y^2 = 2^2$

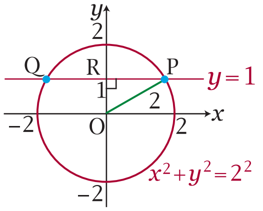
ii) $r = \frac{1}{\sin\theta} \Rightarrow r\sin\theta = 1 \quad \therefore y = 1$

From i) and ii), in the graph,

OP = 2 and OR = 1

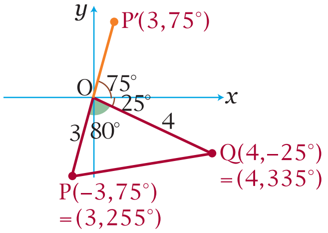
$\therefore PR = \sqrt{2^2 - 1^2} = \sqrt{3}$

$\therefore PQ = 2\sqrt{3}$



Ans (E)

8



If the figure,

$P = (-3, 75^\circ) = (3, 180^\circ + 75^\circ) = (3, 255^\circ)$

$Q = (4, -25^\circ) = (4, 360^\circ - 25^\circ) = (4, 335^\circ)$

$\therefore\ \angle POQ = 335^\circ - 255^\circ = 80^\circ$

In triangle POQ,

$PQ^2 = 3^2 + 4^2 - 2\cdot 3\cdot 4\cos 80^\circ = 25 - 24\cos 80^\circ$

$\therefore\ PQ = \sqrt{25 - 24\cos 80^\circ} = 4.5642\cdots\cdots \approx 4.56$

Ans (B)

Lesson 78 POLAR EQUATIONS

Complex numbers

2 From $z = 2(\cos 50^\circ + i\sin 50^\circ)$,

$\bar{z} = 2(\cos 50^\circ - i\sin 50^\circ)$

$= 2(\cos(-50^\circ) + i\sin(-50^\circ))$

Ans (C)

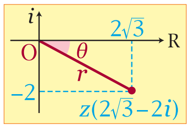
3 From the complex number $z = 2\sqrt{3} - 2i$,

$r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{4\cdot 3 + 4} = 4$

and $\tan\theta = \dfrac{-2}{2\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$ $\quad\therefore\ \theta = -\dfrac{\pi}{6}$

$\therefore\ \theta = 2\pi - \dfrac{\pi}{6} = \dfrac{11}{6}\pi$

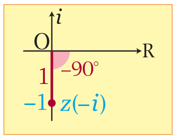
$\therefore\ z = 4\left(\cos\dfrac{11}{6}\pi + i\sin\dfrac{11}{6}\pi\right)$



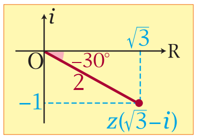
Ans (D)

4 From the complex umber $\dfrac{-i}{\sqrt{3}-i}$,

i)



ii)



i) $-i = \sqrt{0^2 + (-1)^2}(\cos(-90^\circ) + i\sin(-90^\circ))$

$= \cos(-90^\circ) + i\sin(-90^\circ)$

ii) $\sqrt{3} - i$

$= \sqrt{(\sqrt{3})^2 + (-1)^2}(\cos(-30^\circ) + i\sin(-30^\circ))$

$= 2(\cos(-30^\circ) + i\sin(-30^\circ))$

From i) and ii),

$\dfrac{-i}{\sqrt{3}-i} = \dfrac{\cos(-90^\circ) + i\sin(-90^\circ)}{2(\cos(-30^\circ) + i\sin(-30^\circ))}$

$= \dfrac{1}{2}[(\cos(-90^\circ) - (-30^\circ)) + (i\sin(-90^\circ) - (-30^\circ))]$

$= \dfrac{1}{2}(\cos(-60^\circ) + i\sin(-60^\circ))$

Ans (E)

5 Since $x = \cos 15^\circ + i\sin 15^\circ$,

$x^{20} - \dfrac{1}{x^{20}} = x^{20} - x^{-20}$

$= (\cos 15^\circ + i\sin 15^\circ)^{20} - (\cos 15^\circ + i\sin 15^\circ)^{-20}$

$= (\cos 20\cdot 15^\circ + i\sin 20\cdot 15^\circ)$

$\quad - (\cos(-20\cdot 15^\circ) + i\sin(-20\cdot 15^\circ))$

$= (\cos 300^\circ + i\sin 300^\circ)$

$\quad - (\cos(-300^\circ) + i\sin(-300^\circ))$

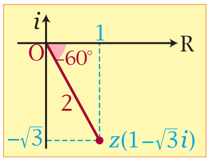
$= (\cancel{\cos 300^\circ} + i\sin 300^\circ) - (\cancel{\cos 300^\circ} - i\sin 300^\circ)$

$= \cos 300^\circ + i\sin 300^\circ - \cos 300^\circ + i\sin 300^\circ$

$= 2i\sin 300^\circ = 2i\sin(-60^\circ) = 2i\cdot\left(-\dfrac{\sqrt{3}}{2}\right) = -\sqrt{3}i$

Ans (E)

6 If the complex number $z = (1 - \sqrt{3}i)^{\frac{3}{2}}$,



$z = \left[\sqrt{1^2 + (-\sqrt{3})^2}(\cos(-60^\circ) + i\sin(-60^\circ))\right]^{\frac{3}{2}}$

$= [2(\cos(-60^\circ) + i\sin(-60^\circ))]^{\frac{3}{2}}$

$= 2^{\frac{3}{2}}\left(\cos\left(\dfrac{3}{2}\cdot(-60^\circ)\right) + i\sin\left(\dfrac{3}{2}\cdot(-60^\circ)\right)\right)$

$= 2\sqrt{2}(\cos(-90^\circ) + i\sin(-90^\circ))$

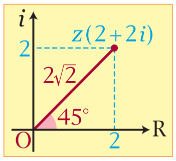
$= 2\sqrt{2}(0 + i(-1)) = -2\sqrt{2}i$

Ans (B)

7 From $z = 2 + 2i$,

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

and $\tan\theta = \dfrac{2}{2} = 1 \quad\therefore\ \theta = 45^\circ$



$\therefore\ (2+2i)^{\frac{1}{3}} = 2\sqrt{2}(\cos 45^\circ + i\sin 45^\circ))^{\frac{1}{3}}$

$= (2\sqrt{2})^{\frac{1}{3}}\left(\cos\dfrac{360^\circ k + 45^\circ}{3} + i\sin\dfrac{360^\circ k + 45^\circ}{3}\right)$

$= \sqrt{2}(\cos(120^\circ k + 15^\circ) + i\sin(120^\circ k + 15^\circ))$

i) $k = 0 \Rightarrow z_1 = \sqrt{2}(\cos 15^\circ + i\sin 15^\circ)$

ii) $k = 1 \Rightarrow z_2 = \sqrt{2}(\cos 135^\circ + i\sin 135^\circ)$

$[\leftarrow \cos(120^\circ + 15^\circ) + i\sin(120^\circ + 15^\circ)]$

iii) $k = 2 \Rightarrow z_3 = \sqrt{2}(\cos 255^\circ + i\sin 255^\circ)$

$[\leftarrow \cos(240^\circ + 15^\circ) + i\sin(240^\circ + 15^\circ)]$

Ans (D)

Since the plane is parallel to the z-axis, the equation of the line is equal to the equation of the plane.

$\therefore -2x + 3y = -6 \quad \therefore 2x - 3y = 6$

Ans (A)

6 From two passing points A(2, –1, 4) and B(1, –3, 6), the equation of the line $\overline{AB}$ is

$$\frac{x-2}{1-2} = \frac{y-(-1)}{-3-(-1)} = \frac{x-4}{6-4}$$

$$\therefore \frac{x-2}{-1} = \frac{y+1}{-2} = \frac{x-4}{2}$$

Since the xy-plane is $z = 0$,

if the intersection point P(a, b, 0),

$$\frac{a-2}{-1} = \frac{b+1}{-2} = \frac{0-4}{2} \ (= -2)$$

$\therefore a - 2 = 2$ and $b + 1 = 4$

$\therefore a = 4$ and $b = 3 \qquad \therefore$ P(4, 3, 0)

Ans (D)

7 From the plane $20x + 15y + 6z = 60$,

the x-intercept $(\leftarrow y = z = 0)$

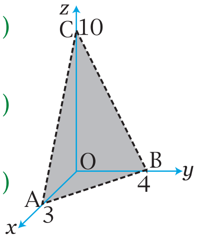
$\Rightarrow 20x = 60 \quad \therefore x = 3$

the y-intercept $(\leftarrow z = x = 0)$

$\Rightarrow 15y = 60 \quad \therefore y = 4$

the z-intercept $(\leftarrow x = y = 0)$

$\Rightarrow 6z = 60 \quad \therefore z = 10$



In the figure, the volume of the pyramid O-ABC is $\frac{1}{3} \times \underbrace{\left(\frac{1}{2}\cdot 3\cdot 4\right)}_{\text{Base}} \times \underbrace{10}_{\text{Height}} = \frac{1}{3}\cdot 6\cdot 10$

$= 20$

Ans (A)

8 From two planes $\begin{cases} 2x + y - z = 13 \cdots\cdots ① \\ x - 2y + z = -4 \cdots\cdots ② \end{cases}$

from ① + ②, $3x - y = 9 \quad \therefore y = 3x - 9 \cdots\cdots ③$

from ③ → ①, $2x + (3x - 9) - z = 13$

$\therefore 5x - z = 22 \quad \therefore z = 5x - 22$

$\therefore (x, 3x - 9, 5x - 22)$

∴ The equation of the line is

$x = t,\ y = 3t - 9,\ z = 5t - 22$

Since point $(a, 3, b)$ lies on the line l,

$a = t,\ 3 = 3t - 9,\ b = 5t - 22$

$$\therefore \begin{cases} 3 = 3a - 9 \quad \therefore 3a = 12 \quad \therefore a = 4 \\ b = 5a - 22 \quad \therefore b = 5\cdot 4 - 22 \quad \therefore b = -2 \end{cases}$$

$\therefore a + b = 4 + (-2) = 2$

Ans (B)

Lesson 85 SPACE

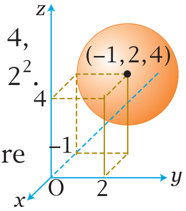
Spheres

2 From the sphere

$(x + 1)^2 + (y - 2)^2 + (z - 4)^2 = 4$,

$(x + 1)^2 + (y - 2)^2 + (z - 4)^2 = 2^2$.

Since its center is (–1, 2, 4) and the radius is 2, the sphere is tangent to the xz-plane.



Ans (E)

3 Since the equation of the sets of the points is the equation of a sphere with radius 4 and the center (–1, 2, –3),

$(x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 4^2$

$\therefore x^2 + 2x + 1 + y^2 - 4y + 4 - z^2 + 6z + 9 = 16$

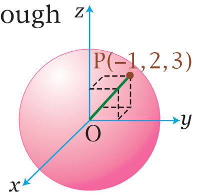
$\therefore x^2 + y^2 + z^2 + 2x - 4y + 6z - 2 = 0$

Ans (E)

4 Since the sphere passes through the point (–1, 2, 3), the radius of the sphere is

$\text{OP} = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$

∴ The equation of the sphere is $x^2 + y^2 + z^2 = 14$



(A) (0, 0, 5) $\Rightarrow 0^2 + 0^2 + 5^2 = 25 > 14$ [← Out]

(B) (0, 2, 4) $\Rightarrow 0^2 + 2^2 + 4^2 = 20 > 14$ [← Out]

(C) (2, –2, 2) $\Rightarrow 2^2 + (-2)^2 + 2^2 = 12 < 14$ [← In]

(D) $(3, \sqrt{5}, 0) \Rightarrow 3^2 + (\sqrt{5})^2 + 0^2 = 14$ [← On]

(E) $(-1, 2, \sqrt{10}) \Rightarrow (-1)^2 + 2^2 + (\sqrt{10})^2$

$= 15 > 14$ [← Out]

∴ The point (2, –2, 2) lies inside the sphere.

Ans (C)

5 From the equation of the sphere

$x^2 + y^2 + z^2 - 2x + 6y - 8z + 1 = 0$,

$x^2 - 2x + y^2 + 6y + z^2 - 8z + 1 = 0$

$\therefore (x-1)^2 + (y+3)^2 + (z-4)^2 = 1 + 9 + 16 - 1$

$\therefore (x-1)^2 + (y+3)^2 + (z-4)^2 = 5^2$

Since the radius of the sphere is 5, its volume is $\frac{4}{3}\pi \cdot 5^3 = 523.5987\cdots\cdots \approx 523.6$

Ans (E)

6 Since A(–3, 4, 2) and B(1, 2, –2) are the endpoints of diameter, the center of the sphere is the midpoint of A and B.

$\therefore C\left(\frac{-3+1}{2}, \frac{4+2}{2}, \frac{2+(-2)}{2}\right) = (-1, 3, 0)$

Since the radius of the sphere is the length of $\overline{AC}$, $r = \sqrt{((-1)-(-3))^2 + (3-4)^2 + (0-2)^2}$

$= \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$

$\therefore$ The equation of the sphere is

$(x+1)^2 + (y-3)^2 + z^2 = 3^2$

$\therefore x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 - 9 = 0$

$\therefore x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$

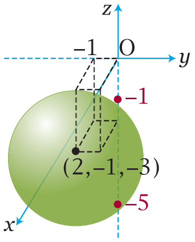
Ans (A)

7 Since the z-axis is $x = y = 0$, the z-intercepts of the sphere $(x-2)^2 + (y+1)^2 + (z+3)^2 = 3^2$ are $(-2)^2 + 1^2 + (z+3)^2 = 9$.

$\therefore (z+3)^2 = 4 \quad \therefore z + 3 = \pm 2$

$\therefore z = \pm 2 - 3 = -1, -5$

$\therefore$ The distance of the two z-intercepts is $(-1) - (-5) = 4$.



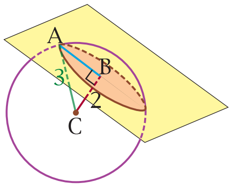
Ans (B)

8 From the sphere $x^2 + y^2 + z^2 - 2x - 2y = 7$,

$x^2 - 2x + y^2 - 2y + z^2 = 7$,

$(x-1)^2 + (y-1)^2 + z^2 = 9$

$\therefore$ The center of the sphere C(1, 1, 0) and the radius CA = 3



$\therefore$ The distance of the center C(1, 1, 0) of the sphere and the plane $2x + y + 2z = 9$

is $\frac{|2\cdot1 + 1 + 2\cdot0 - 9|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{|3-9|}{3} = 2$

$\therefore$ The radius of the circle AB $= \sqrt{3^2 - 2^2} = \sqrt{5}$

$\therefore$ The area of the circle is $(\sqrt{5})^2\pi = 5\pi$

$= 15.708\cdots\cdots \approx 15.71$

Ans (D)

Lesson
85
86

Lesson 86 COUNTING

Permutations

2 $\frac{(n+1)! - n!}{(n-1)!} = \frac{n!(n+\cancel{1}-\cancel{1})}{(n-1)!} = \frac{n\cdot n!}{(n-1)!}$

$= \frac{n\cdot n\cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n^2$

Ans (E)

3 $\frac{(3!)!}{(3!)^2} = \frac{(3\cdot2\cdot1)!}{(3\cdot2\cdot1)^2} = \frac{6!}{6^2}$

$= \frac{\cancel{6}\cdot5\cdot4\cdot3\cdot2\cdot1}{\cancel{6}\cdot6} = \frac{5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot1}{\cancel{6}}$

$= 5\cdot4 = {}_5P_2 = {}_nP_r \quad \therefore n = 5$ and $r = 2$

$\therefore n + r = 5 + 2 = 7$

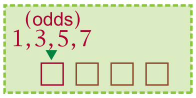
Ans (C)

4 From the seven digits 1, 2, 3, 4, 5, 6 and 7,

$\underset{\text{Odds}}{{}_4P_1} \times {}_{7-1}C_3 \times 3!$

$= 4 \times {}_6C_3 \times 3! = 4 \times {}_6P_3$

$= 4 \times 6\cdot5\cdot4 = 4 \times 120 = 480$



Ans (D)

5 From the six digits 0, 1, 2, 3, 4 and 5,

i) $1\,\square\,\square\,\underset{0,2,4}{\square} \Rightarrow \underset{\text{Even}}{\underline{{}_3C_1}} \times {}_4P_2 = 3 \times (4\times3) = \underset{\text{36th}}{\underline{36}}$

ii) $2\,\square\,\square\,\underset{0,4}{\square} \Rightarrow \underset{\text{Even}}{\underline{{}_2C_1}} \times {}_4P_2 = 3 \times (4\times3) = \underset{\text{72nd}}{\underline{36}}$

iii) $3\;0\,\square\,\underset{2,4}{\square} \Rightarrow \underset{\text{Even}}{\underline{{}_2C_1}} \times {}_3P_1 = 2 \times 3 = \underset{\text{78th}}{\underline{6}}$

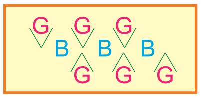
iv) $\underset{\text{79th}}{3102} < \underset{\text{80th}}{3104} < \underset{\text{81th}}{3120}\,,\ \underset{\text{82th}}{3124} < \underset{\text{83th}}{3140}$

Ans (B)

6 From 3 boys and 3 girls,

i) G B G B G B $\Rightarrow \underset{\text{girls first}}{3!} \times \underset{\text{boys}}{3!}$

ii) B G B G B G $\Rightarrow \underset{\text{boys first}}{3!} \times \underset{\text{girls}}{3!}$



From i) and ii),

$2 \times (3! \times 3!) = 2 \times (3\cdot2\cdot1) \times (3\cdot2\cdot1)$

$= 2 \times 6\cdot6 = 72$

Ans (C)

7 i) Choosing one color for the bottom face.

$\Rightarrow {}_6C_1 = 6$

ii) Arranging the remaining five colors for