Chapter 16 – Probability

FREE FAST TRACK LECTURES:

https://www.youtube.com/watch?v=ZZpvRpkgmaE&list=PLAKrxMrPL3fwOSJWxnr8j0C9a4si2Dfdz

Link for Lecture 1 of Probability (Premiering on 19-06-2020) at 10:00 a.m.:
https://youtu.be/5W4ylV61o20

Link for Lecture 2 of Probability (Premiering on 20-06-2020) at 10:00 a.m.:
https://youtu.be/36XjUgvbtfs

Introduction

The result of a random experiment is known as an event or an outcome. Probability is the chance of an outcome.

$$\text{Probability} = \frac{\text{No. of Favourable Cases/Events/Outcomes}}{\text{Total No. of Cases/Events/Outcomes}}$$

Question 1

A coin is tossed. What is the probability that the outcome will be heads?

Solution

Total no. of outcomes = 2 (i.e., Heads, and Tails)

No. of favourable outcomes = 1 (i.e., heads)

Therefore, $\text{Probability} = \frac{1}{2}$

Question 2

A dice is rolled. Find the probability for the following outcomes:

1. Number 1 appears
2. Number 4 appears
3. Even number appears
4. Odd number appears
5. Number greater than 4 appears

Solution

Total number of outcomes = 6 (i.e., 1, 2, 3, 4, 5, 6)

1. Number 1 appears
   No. of favourable outcomes = 1 (i.e., 1)

Therefore, Probability $= \frac{1}{6}$

2. Number 4 appears
   No. of favourable outcomes = 1 (i.e., 4)
   Therefore, Probability $= \frac{1}{6}$
3. Even number appears
   No. of favourable outcomes = 3 (i.e., 2, 4, 6)
   Therefore, Probability $= \frac{3}{6} = \frac{1}{2}$
4. Odd number appears
   No. of favourable outcomes = 3 (i.e., 1, 3, 5)
   Therefore, Probability $= \frac{3}{6} = \frac{1}{2}$
5. Number greater than 4 appears
   No. of favourable outcomes = 2 (i.e., 5, 6)
   Therefore, Probability $= \frac{2}{6} = \frac{1}{3}$

Equally Likely Events or Mutually Symmetric Events or Equi-Probable Events

If two or more events have the same probability, the events are said to be equally likely events. For example, when a coin is tossed, the probability of getting heads is ½; also, the probability of getting tails is also ½. Therefore, these two events are said to be equally likely.

Similarly, when a dice is thrown, the probability of getting either 1, or 2, or 3, or 4, or 5, or 6 is 1/6. Therefore, these events are known as equally likely events.

The events which have different probabilities are said “Not Equally Likely” events.

Impossible Events

Events which have zero probability are known as “Impossible Events”. For example, let today be Monday. Now, the probability that tomorrow is going to be Wednesday is zero. Therefore, this event is an impossible event.

Sure/Certain Events

Events which have 100% (or 1) probability are known as “Sure/Certain Events”. For example, let today be Wednesday. Now, the probability that tomorrow is going to be Thursday is 100%, i.e. 1. Therefore, this event is a sure/certain event.

From the above discussion on impossible and certain events, it can be seen that the probability ranges from 0 to 1 (both inclusive). Probability can never be a negative number.

Mutually Exclusive Events or Incompatible Events

The events which cannot occur simultaneously are called mutually exclusive events. For example, when a coin is tossed, there are a total of two outcomes – Heads, and Tails. However, these two events cannot occur at the same time. If heads occur, tails would not occur; and if tails occur, heads would not occur. Therefore, these two events are said to be mutually exclusive events. "Mutually exclusive events" is technically defined as follows: when the occurrence of one event prevents the occurrence of other event, such events are known as mutually exclusive events.

The events which can occur simultaneously are called "Not Mutually Exclusive" events. For example, when a dice is rolled, the events "odd number occurs", and "number 5 occurs" can occur together. Therefore, these events are called "not mutually exclusive" events.

Sample Space

The set of all possible events is known as Sample Space. The sum of probabilities of every element in the sample space is always 1. For example, when a coin is tossed, two events may occur – Heads, or Tails. Therefore, the sample space is S: $\{H, T\}$. Now, the probability of heads is ½, and the probability of tails is ½, and the sum total of this is ½ + ½ = 1. Therefore, we see that the probability of the sample space is 1. Similarly, when a dice is rolled, 6 events may occur – 1, 2, 3, 4, 5, or 6. Therefore, the sample space is S: {1, 2, 3, 4, 5, 6}. Now, the probability of each of these events is 1/6. Sum total of the probability of all the events is 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1. Therefore, again, we see that the probability of the sample space is 1.

Simple (or Elementary) and Composite (or Compound) Events

An event which cannot be split into two or more parts is known as a simple event. For example, when a dice is thrown, the event "5 occurs" cannot be broken down into any more parts.

An event which can be broken down into two or more simple events is known as a composite event. For example, when a dice is thrown, the event "odd number occurs" can be broken down into two or more parts. This is because, if the numbers 1, 3, or 5 occur, they correspond to our event "odd number occurs". Therefore, the event "odd number occurs" can be broken down into 3 parts – "1 occurs", "3 occurs", and "5 occurs". Similarly, on throwing of a dice, the event "number more than 2 occurs" can be split into 4 parts – "3 occurs", "4 occurs", "5 occurs", and "6 occurs". Therefore, the event "number more than 2 occurs" is also a composite event.

Exhaustive Events

If a coin is tossed, there are two possible events – Heads, or Tails. Now, one of these events will necessarily occur when a coin is tossed. Such events are known as exhaustive events.

Classical Definition of Probability or a Prior Definition

Consider a random experiment with total number of outcomes denoted by n. Consider the event denoted by A for which we have to find out the probability. Suppose the number of outcomes in favour of event A is denoted by n_A. Then Probability of event A is given by:

$$P(A)=\frac{n_A}{n}=\frac{\text{No. of equally likely events favourable to } A}{\text{Total no. of equally likely events}}$$

Now, instead of considering all the events, let's consider only the mutually exclusive, exhaustive, and equally likely events, denoted by m. Consider the event denoted by A for which we have to find out the probability. Suppose the number of mutually exclusive, exhaustive, and equally likely outcomes in favour of event A is denoted by m_A. Then Probability of event A is given by:

$$P(A)=\frac{m_A}{m}=\frac{\text{No. of mutually exclusive, exhaustive, and equally likely events favourable to } A}{\text{Total no. of mutually exclusive, exhaustive, and equally likely events}}$$

Limitations or Demerits of Classical Definition of Probability

1. It is applicable only when the total no. of events is finite.
2. It can be used only when the events are equally likely or equi-probable.
3. This definition has only a limited field of application like
   a. coin tossing,
   b. dice throwing,
   c. drawing cards etc.

   where the possible events are known well in advance.
   In the field of uncertainty or where no prior knowledge is provided, this definition is inapplicable.

Properties

1. The probability of an event lies between 0 and 1, both inclusive, i.e. $0 \le P(A) \le 1$.

   When $P(A)=0$, A is known as an impossible event, and when $P(A)=1$, A is known as a sure/certain event.
2. Non-occurrence of event A is denoted by A', or A^c, or $\bar{A}$, and it is known as complimentary event of A. (This is similar to the compliment of a set that we studied in set theory.) The event A along with its complimentary A' forms a set of mutually exclusive and exhaustive events.

   $P(A)+P(A')=1$

   $\Rightarrow P(A')=1-P(A)=1-\frac{m_A}{m}=\frac{m-m_A}{m}$

   Therefore, $P(A')=\frac{m-m_A}{m}$
3. The ratio of no. of favourable events to the no. of unfavourable events is known as odds in favour of the event A and its inverse ratio is known as odds against the event A.

   Odds in favour of $A \rightarrow m_A:(m-m_A)$

   Odds against $A \rightarrow (m-m_A):m_A$
4. Computation of total number of outcomes when an experiment is repeated a certain number of times:
   When an experiment with total number of events a is repeated b number of times, the total number of outcomes is given by a^b.

Question 3

A coin is tossed three times. What is the probability of getting:

1. 2 heads
2. At least 2 heads

Solution

Here, the experiment is tossing of a coin, containing 2 outcomes – Heads and Tails. This experiment is repeated 3 times. Therefore, total number of outcomes is $2^3 = 8$.

The sample space for the given experiment is *S*: {(*HHH*), (*HHT*), (*HTH*), (*HTT*), (*THH*), (*THT*), (*TTH*), (*TTT*)}

1. Probability of 2 heads
   We can see that no. of events containing two heads are 3, i.e., *HHT*, *HTH*, *THH*.
   Therefore, Probability = 3/8 = 0.375.
2. Probability of at least 2 heads
   We can see that no. of events containing at least two heads are 4, i.e., *HHH*, *HHT*, *HTH*, *THH*.
   Therefore, Probability = 4/8 = 0.5.

Question 4

A dice is rolled twice. What is the probability of getting a difference of 2 points?

Solution

Here, the experiment is rolling of a dice, containing 6 outcomes. This experiment is repeated twice. Therefore, total number of outcomes is $6^2 = 36$.

Events in which the difference is of 2 points are {(1, 3), (2, 4), (3, 5), (4, 6), (3, 1), (4, 2), (5, 3), (6, 4)}. Therefore, total number of outcomes in favour of the event = 8.

Therefore, probability = 8/36 = 0.22.

Question 5

Two dice are thrown simultaneously. Find the probability that the sum of points on the two dice would be 7 or more.

Solution

Two dice are thrown simultaneously is the same as one dice being thrown twice. Therefore, the total number of outcomes are $6^2 = 36$.

The question asks the probability of the sum being 7 or more. We know that the highest sum can be 12 when both the dice show 6. Now, a total of 7 or more, i.e., 7, or 8, or 9, or 10, or 11, or 12 can occur in the following combinations:

Condition	Events Corresponding to Condition	Total Events
Sum of 7	{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}	6

Sum of 8	{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}	5
Sum of 9	{(3, 6), (4, 5), (5, 4), (6, 3)}	4
Sum of 10	{(4, 6), (5, 5), (6, 4)}	3
Sum of 11	{(5, 6), (6, 5)}	2
Sum of 12	{(6, 6)}	1
Total		21

Therefore, probability = 21/36 = 0.58.

Question 6

What is the chance of picking a spade or an ace not of spade from a pack of 52 cards?

(a) 4/13 (b) 5/13 (c) 6/13 (d) 7/13

Solution

We know that in a pack of cards, we have:

Heart	:	2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace	=	13 Cards
Diamond	:	2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace	=	13 Cards
Spade	:	2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace	=	13 Cards
Club	:	2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace	=	13 Cards
Total				52 Cards

Therefore, probability of picking a spade = 13/52 = 0.25

Also, probability of picking an ace not of spade = 3/52 = 0.06

Therefore, probability of picking a spade ***or*** an ace not of spade = 0.25 + 0.06 = 0.31

Try the options:

Option (a) → 4/13

On calculator, we can see that 4/13 = 0.31. Therefore, option (a) is the answer.

Question 7

Find the probability that a four-digit number comprising the digits 2, 5, 6 and 7 would be divisible by 4.

Solution

Total number of 4 digit numbers that can be formed from these 4 digits = 4! = 24.

Any number is divisible by four if the number formed by the last two digits of that number is divisible by 4. For example, consider the number 45620. Now, the last two digits of this number are 2, and 0. The number formed from these two digits is 20, which is divisible by 4. Therefore, the number 45620 is also divisible by 4.

Now, we have 2, 5, 6, and 7. The two digit numbers that can be formed from these digits which are divisible by 4 are 52, 56, 72, and 76. Therefore, we know that the last two digits could either be 52, 56, 72, and 76. Therefore, there are 4 ways to fill the last two digits of a four-digit number.

Now, after filling the last two spaces, we'll be left with the first two spaces and two digits to fill them. Hence, the first two digits can be filled in 2! ways.

Therefore, the no. of 4 digit numbers that can be formed from the digits 2, 5, 6, and 7, which are divisible by 4 is 2! × 4 = 8.

Therefore, probability = 8/24 = 0.33.

Question 8

A committee of 7 members is to be formed from a group comprising 8 gentlemen and 5 ladies. What is the probability that the committee would comprise of:

1. 2 ladies,
2. At least 2 ladies.

(a) 140/429; 392/429 (b) 147/435; 392/429 (c) 140/429; 399/429 (d) 140/429; 392/478

Solution

There are 8 + 5 = 13 people in total. A committee of 7 members can be formed in ${}^{13}C_7 = \frac{13!}{7!(13-7)!} = \frac{13!}{7!6!}$ = 1,716 ways.

1. Committee comprising of two ladies
   No. of ways a committee can be formed consisting of 2 ladies = ${}^{5}C_2 \times {}^{8}C_5$ = 560 ways.
   Therefore, the probability that the committee would comprise of 2 ladies = 560/1716 = 0.3263.
2. Committee comprising of at least two ladies
   No. of ways a committee can be formed comprising at least two ladies:

Combinations	Ladies	Gents		Total
2 ladies + 5 gents	${}^{5}C_2$	${}^{8}C_5$	${}^{5}C_2 \times {}^{8}C_5$	560
3 ladies + 4 gents	${}^{5}C_3$	${}^{8}C_4$	${}^{5}C_3 \times {}^{8}C_4$	700
4 ladies + 3 gents	${}^{5}C_4$	${}^{8}C_3$	${}^{5}C_4 \times {}^{8}C_3$	280
5 ladies + 2 gents	${}^{5}C_5$	${}^{8}C_2$	${}^{5}C_5 \times {}^{8}C_2$	28
Total				1,568

   Therefore, the probability that the committee would comprise at least of 2 ladies = 1568/1716 = 0.9138

Now, try the options:

Option (a) → 140/429; 392/429

On calculator, 140/429 = 0.3263; and 392/429 = 0.9138. Therefore, option (a) is the answer.

Relative Frequency Definition of Probability

This is self-explanatory. Look at the question below:

Question 9

The following data relates to the distribution of wages of a group of workers:

Wages in ₹	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100	100 – 110	110 – 120
No. of workers	15	23	36	42	17	12	5

If a worker is selected at random from the entire group of workers, what is the probability that

1. his wage would be less than ₹50?
2. his wage would be less than ₹80?
3. his wage would be more than ₹100?
4. his wages would be between ₹70 and ₹100?

Solution

Total no. of workers = 15 + 23 + 36 + 42 + 17 + 12 + 5 = 150

1. Since there is no worker whose wage is less than ₹50, therefore, the probability is 0.
2. No. of workers whose wages are less than ₹80 = 15 + 23 + 36 = 74
   Therefore, probability = 74/150 = 0.49
3. No. of workers whose wages are more than ₹100 = 12 + 5 = 17
   Therefore, probability = 17/150 = 0.11
4. No. of workers whose wages are between ₹70 and ₹100 = 36 + 42 + 17 = 95
   Therefore, probability = 95/150 = 0.63

Operations on Events – Set Theoretic Approach to Probability

Sample space represents the Universal Set, denoted by *S* or Ω. An event *A* is defined as a non-empty subset of *S*. Then, probability of event *A* is given by: $P(A)=\dfrac{n(A)}{n(S)}$, where, $n(A)$ is the cardinal number of the set *A*; and $n(S)$ is the cardinal number of the set *S*.

Points to be Noted

1. Two events *A* and *B* are mutually exclusive, if $A\cap B=\phi$. Therefore, $P(A\cap B)=0$, or $P(A\cup B)=P(A)+P(B)$. Similarly, three events *A*, *B*, and *C* are mutually exclusive, if $P(A\cup B\cup C)=P(A)+P(B)+P(C)$.
2. Two events *A* and *B* are exhaustive, if $P(A\cup B)=1$. Similarly, three events *A*, *B*, and *C* are exhaustive, if $P(A\cup B\cup C)=1$.
3. Three events *A*, *B*, and *C* are equally likely if $P(A)=P(B)=P(C)$.

Question 10

Three events *A*, *B* and *C* are mutually exclusive, exhaustive and equally likely. What is the probability of the complementary event of *A*?

Solution

Since all three events are mutually exclusive, we have

$P(A\cup B\cup C)=P(A)+P(B)+P(C)$ … Eq. (1)

Also, since all three events are exhaustive, we have

$P(A\cup B\cup C)=1$ … Eq. (2)

Again, since all three events are equally likely, we have

$P(A)=P(B)=P(C)$

Let $P(A)=P(B)=P(C)=k$ … Eq. (3)

Combining equations (1) and (2), we have

$P(A)+P(B)+P(C)=1$ … Eq. (4)

Combining equations (3) and (4), we have

$k+k+k=1$

Therefore, $3k=1\Rightarrow k=\frac{1}{3}$

Therefore, $P(A)=P(B)=P(C)=\frac{1}{3}$

We know that $P(A)+P(A')=1$

Therefore, $P(A')=1-P(A)=1-\frac{1}{3}=\frac{2}{3}$

Axiomatic or Modern Definition of Probability

Important Theorems –

1. For any two **mutually exclusive** events *A* and *B*, the probability that either *A* or *B* occurs is given by the sum of individual probabilities of A and B.
   Consider two mutually exclusive events *A* and *B*. The probability that either *A* or *B* occurs is given by $P(A\cup B)=P(A)+P(B)$.
   Refer Question 11.
2. For any $k\geq 2$ **mutually exclusive** events $A_1, A_2, A_3, ..., A_k$, the probability that at least one of them occurs is given by the sum of individual probabilities of the k events.
   $P(A_1\cup A_2\cup A_3\cup ...\cup A_k)=P(A_1)+P(A_2)+P(A_3)+...+P(A_k)$
3. For any two events *A* and *B*, the probability that either *A* or *B* occurs is given by the sum of individual probabilities of *A* and *B* less the probability of simultaneous occurrence of the events *A* and *B*.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Refer Question 12.

4. For any three events *A*, *B* and *C*, the probability that at least one of the events occurs is given by $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

   Refer Question 15.

Question 11

A number is selected from the first 25 natural numbers. What is the probability that it would be divisible by 4 or 7?

Solution

We have $n(S) = 25$.

Let *A* be the event that the number is divisible by 4, and *B* be the event that the number is divisible by 7.

Let's find out if there's any number which is divisible by both 4, as well as 7. The LCM of 4 and 7 is 28. Therefore, any number divisible by 28 is divisible by 4 as well as 7. However, our sample space contains numbers only up to 25. Therefore, in our sample space, there cannot be any number which is divisible by 4 as well as by 7. Hence, the events *A* and *B* are mutually exclusive. Therefore, we'll use the formula: $P(A \cup B) = P(A) + P(B)$.

We have *A* = {4, 8, 12, 16, 20, 24}. Therefore, $n(A) = 6$

We have *B* = {7, 14, 21}. Therefore, $n(B) = 3$

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{6}{25}$

$P(B) = \frac{n(B)}{n(S)} = \frac{3}{25}$

Therefore, probability that either *A* or *B* occurs $[P(A \cup B)] = P(A) + P(B) = \frac{6}{25} + \frac{3}{25} = \frac{9}{25}$

Question 12

A number is selected at random from the first 1000 natural numbers. What is the probability that it would be a multiple of 5 or 9?

Solution

We have $n(S) = 1000$.

If a number is multiple of 5, that means it is divisible by 5. If a number is a multiple of 9, that means it is divisible by 9.

Let A be the event that the number is divisible by 5.

Let B be the event that the number is divisible by 9.

Let's find out if there's any number which is divisible by both 5, as well as 9. The LCM of 5 and 9 is 45. Therefore, any number divisible by 45 is divisible by 5 as well as 9. Our sample space contains numbers up to 1,000. Therefore, in our sample space, there will be many numbers which are divisible by 5 as well as by 9. Therefore, the events A and B are NOT mutually exclusive. Therefore, we'll use the formula: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$(A \cap B)$ denotes the event that the number is divisible by 5 as well as by 9.

Therefore,

No. of numbers divisible by $5 = \frac{1000}{5} = 200 \Rightarrow n(A) = 200$

No. of numbers divisible by $9 = \frac{1000}{9} = 111.11 = 111 \Rightarrow n(B) = 111$

No. of numbers divisible both by 5 and $9 = \frac{1000}{45} = 22.22 = 22 \Rightarrow n(A \cap B) = 22$

Therefore,

$$P(A) = \frac{n(A)}{n(S)} = \frac{200}{1000} = 0.20$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{111}{1000} = 0.111$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{22}{1000} = 0.022$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.20 + 0.111 - 0.022 = 0.289$$

Therefore, the probability that the number is a multiple of either 5 or 9 is 0.289.

Question 13

The probability that an Accountant's job applicant has a B. Com. Degree is 0.85, that he is a CA is 0.30 and that he is both B. Com. and CA is 0.25. Out of 500 applicants, how many would be B. Com. or CA?

Solution

We have $n(S) = 500$.

Let B be the event that the applicant has a B. Com degree.

Let C be the event that the applicant has a CA degree.

Then, $(B \cap C)$ is the event that the applicant has both B. Com as well as CA degree.

Therefore, $P(B) = 0.85$; $P(C) = 0.30$; and $P(B \cap C) = 0.25$.

$$P(B \cup C) = P(B) + P(C) - P(B \cap C)$$

$$P(B \cup C) = 0.85 + 0.30 - 0.25 = 0.90$$

Also,

$$P(B \cup C) = \frac{n(B \cup C)}{n(S)} \Rightarrow 0.90 = \frac{n(B \cup C)}{500} \Rightarrow n(B \cup C) = 500 \times 0.90 = 450$$

Therefore, number of applicants who have B. Com as well as CA degree = 450. This is also known as Expected Frequency.

Question 14

If $P(A-B) = \frac{1}{5}$, $P(A) = \frac{1}{3}$, and $P(B) = \frac{1}{2}$, what is the probability that out of the two events A and B, only B would occur?

(a) 11/30 (b) 12/30 (c) 13/30 (d) 14/30

Solution

Probability that only B occurs is denoted by $P(B-A)$, and is given by $P(B) - P(B \cap A)$.

Similarly, probability that only A occurs is denoted by $P(A-B)$, and is given by $P(A) - P(A \cap B)$.

Therefore, we have $P(A-B) = P(A) - P(A \cap B)$

$$\Rightarrow P(A \cap B) = P(A) - P(A-B) = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}$$

Now, $P(B-A) = P(B) - P(B \cap A)$

$$\Rightarrow P(B-A) = \frac{1}{2} - \frac{2}{15} = 0.37 \qquad [\text{Since } P(A \cap B) = P(B \cap A)]$$

Now, try the options:

Option (a) $\rightarrow$ 11/30

On calculator, 11/30 = 0.37. Therefore, option (a) is the answer.

Question 15

There are three persons *A*, *B* and *C* having different ages. The probability that *A* survives another 5 years is 0.80, *B* survives another 5 years is 0.60 and *C* survives another 5 years is 0.50. The probabilities that *A* and *B* survive another 5 years is 0.46, *B* and *C* survive another 5 years is 0.32 and *A* and *C* survive another 5 years 0.48. The probability that all these three persons survive another 5 years is 0.26. Find the probability that at least one of them survives another 5 years.

Solution

We have $P(A) = 0.80$; $P(B) = 0.60$; $P(C) = 0.50$; $P(A \cap B) = 0.46$; $P(B \cap C) = 0.32$; $P(A \cap C) = 0.48$; $P(A \cap B \cap C) = 0.26$

We know that $P(A \cup B \cup C)$ = $P(A)$ + $P(B)$ + $P(C)$ − $P(A \cap B)$ − $P(A \cap C)$ − $P(B \cap C)$ + $P(A \cap B \cap C)$

Therefore, $P(A \cup B \cup C) = 0.80 + 0.60 + 0.50 - 0.46 - 0.32 - 0.48 + 0.26 = 0.90$

Conditional Probability and Compound Theorem of Probability

Independent and Dependent Events

Consider an example:

A bag consists of 5 red balls and 4 white balls. Let *A* be the event that a red ball is drawn, and *B* be the event that a white ball is drawn. Now, a ball is drawn. The probability of event *A* (i.e., a red ball is drawn) is 5/9. Now, the ball is put back in the bag, and once again, a ball is drawn. Now, the probability of event *B* (i.e., a white ball is drawn) is 4/9.

Now, let's start with the experiment all over again. A bag consists of 5 red balls and 4 white balls. Let *A* be the event that a red ball is drawn, and *B* be the event that a white ball is drawn. Now, a ball is drawn. The probability of event *B* (i.e., a white ball is drawn) is 4/9. Now, the ball is put back in the bag, and once again, a ball is drawn. Now, the probability of event *A* (i.e., a red ball is drawn) is 5/9.

We can see above that whether we calculate the probability of red ball first, or the white ball first, it doesn't make any difference. In other words, the probability of one event is not getting affected because of the happening of another event. Such events are known as independent events.

Now, consider another example:

A bag consists of 5 red balls and 4 white balls. Let *A* be the event that a red ball is drawn, and *B* be the event that a white ball is drawn. Now, a ball is drawn. The probability of event *A* (i.e., a red ball is drawn) is 5/9. Now, another ball is drawn without putting the first ball back in. So, in total, there are 8 balls in the bag now. Hence, the probability of event *B* (i.e., a white ball is drawn) is 4/8.

Now, let's start with the experiment all over again. A bag consists of 5 red balls and 4 white balls. Let *A* be the event that a red ball is drawn, and *B* be the event that a white ball is drawn. Now, a ball is drawn. The probability of event *B* (i.e., a white ball is drawn) is 4/9. Now, another ball is drawn without putting the first ball back in. So, in total, there are 8 balls in the bag now. Hence, the probability of event *A* (i.e., a red ball is drawn) is 5/8.

We can see above that the probability of the events is getting affected because of the happening of another event. When red ball was drawn first, its probability was 5/9, but when it was drawn second, its probability became 5/8. Similarly, when white ball was drawn first, its probability was 4/9, but when it was drawn second, its probability became 4/8. Such events are known as dependent events.

Rules of Probability When Events are Independent

1. $P(A\cap B)=P(A)\times P(B)$
2. Probability of event *A* given that event *B* has already occurred is given by *P*(*A*/*B*):
   $P(A/B)=\dfrac{P(A\cap B)}{P(B)}$
3. Probability of event *B* given that event *A* has already occurred is given by *P*(*B*/*A*):
   $P(B/A)=\dfrac{P(B\cap A)}{P(A)}$
4. $P(A'/B)=\dfrac{P(A'\cap B)}{P(B)}=\dfrac{P(B)-P(A\cap B)}{P(B)}$
5. $P(A/B')=\dfrac{P(A\cap B')}{P(B')}=\dfrac{P(A)-P(A\cap B)}{1-P(B)}$
6. $P(A'/B')=\dfrac{P(A'\cap B')}{P(B')}=\dfrac{P(A\cup B)'}{1-P(B)}=\dfrac{1-P(A\cup B)}{1-P(B)}$
7. Probability that only event *A* occurs: $P(A-B)=P(A)-P(A\cap B)$
8. Probability that only event *B* occurs: $P(B-A)=P(B)-P(A\cap B)$
9. Probability that only event *A* or only event *B* occurs: $P(A)+P(B)-2P(A\cap B)$

Question 16

Rupesh is known to hit a target in 5 out of 9 shots whereas David is known to hit the same target in 6 out of 11 shots. What is the probability that the target would be hit once they both try?

Solution

Let *A* be the event that Rupesh hits the target.

Let *B* be the event that David hits the target.

Then, $P(A)=\dfrac{5}{9}$; and $P(B)=\dfrac{6}{11}$

Since both are independent events, $P(A \cap B) = P(A) \times P(B)$

Therefore, $P(A \cap B) = \frac{5}{9} \times \frac{6}{11} = \frac{30}{99}$

Now, the probability that the target would be hit by at least one of them is given by $P(A \cup B)$.

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Therefore, $P(A \cup B) = \left\{\frac{5}{9} + \frac{6}{11}\right\} - \frac{30}{99} = 0.80$

Question 17

A pair of dice is thrown together and the sum of points of the two dice is noted to be 10. What is the probability that one of the two dice has shown the point 4?

Solution

Since the total is 10, the sample space is: S = {(4, 6), (5, 5), (6, 4)}. Therefore, $n(S)$ = 3. Out of this, there are two events, which have 4 in one of the dice. Therefore, probability = 2/3.

Question 18

In a group of 20 males and 15 females, 12 males and 8 females are service holders. What is the probability that a person selected at random from the group is a service holder given that the selected person is a male?

Solution

We have $n(S) = 20 + 15 = 35$

Let A be the event that the person is a service holder.

Let B be the event that the person is a male.

Therefore, $P(A) = \frac{12+8}{35} = \frac{20}{35}$; $P(B) = \frac{20}{35}$

Also,

We have to determine the probability of event A given that event B has already occurred. Therefore, we have to find out $P(A/B)$.

We know that $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

$n(A \cap B)$ = male service holders = 12

Therefore, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{12}{35}$

Therefore, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{12/35}{20/35} = \dfrac{12}{20} = 0.60$

Alternatively,

There are 20 males in total, and 12 males are service holders. Therefore, probability that the person selected at random is a male and a service holder is 12/20 = 0.60.

Question 19

In connection with a random experiment, it is found that $P(A) = \dfrac{2}{3}$, $P(B) = \dfrac{3}{5}$, and $P(A \cup B) = \dfrac{5}{6}$. Evaluate the following probabilities:

1. $P(A/B)$
2. $P(B/A)$
3. $P(A'/B)$
4. $P(A/B')$
5. $P(A'/B')$

Solution

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Therefore, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \dfrac{2}{3} + \dfrac{3}{5} - \dfrac{5}{6} = \dfrac{13}{30}$

Now,

1. $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{13/30}{3/5} = \dfrac{13}{18}$
2. $P(B/A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{13/30}{2/3} = \dfrac{13}{20}$
3. $P(A'/B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{\frac{3}{5} - \frac{13}{30}}{\frac{3}{5}} = \dfrac{5}{18}$

4. $P(A/B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)} = \dfrac{\frac{2}{3} - \frac{13}{30}}{1 - \frac{3}{5}} = \dfrac{7}{12}$

5. $P(A'/B') = \dfrac{P(A' \cap B')}{P(B')} = \dfrac{P(A \cup B)'}{1 - P(B)} = \dfrac{1 - P(A \cup B)}{1 - P(B)} = \dfrac{1 - \frac{5}{6}}{1 - \frac{3}{5}} = \dfrac{5}{12}$

Question 20

The odds in favour of an event is 2 : 3 and the odds against another event is 3 : 7. Find the probability that only one of the two events occur.

Solution

Let the two events be denoted by *A* and *B*.

Let m_A denote the no. of events in favour *A*, m_B denote the no. of events in favour of *B*, and m denote the total number of events.

Therefore, $P(A) = \dfrac{m_A}{m}$, and $P(B) = \dfrac{m_A}{m}$

Also, odds in favour of event *A* = $m_A : (m - m_A)$

Therefore, $\dfrac{m_A}{m - m_A} = \dfrac{2}{3} \Rightarrow 3m_A = 2(m - m_A) \Rightarrow 3m_A = 2m - 2m_A \Rightarrow 5m_A = 2m \Rightarrow \dfrac{m_A}{m} = \dfrac{2}{5}$

Therefore, $P(A) = \dfrac{m_A}{m} = \dfrac{2}{5}$

Also, odds against event *B* = $(m - m_B) : m_B$

Therefore, $\dfrac{m - m_B}{m_B} = \dfrac{3}{7} \Rightarrow 7(m - m_B) = 3m_B \Rightarrow 7m - 7m_B = 3m_B \Rightarrow 10m_B = 7m \Rightarrow \dfrac{m_B}{m} = \dfrac{7}{10}$

Therefore, $P(B) = \dfrac{m_B}{m} = \dfrac{7}{10}$

Since the events are independent, we have $P(A \cap B) = P(A) \times P(B) = \dfrac{2}{5} \times \dfrac{7}{10} = \dfrac{14}{50}$.

Probability that only one of the events occur: $P(A) + P(B) - 2P(A \cap B) = \dfrac{2}{5} + \dfrac{7}{10} - \left(2 \times \dfrac{14}{50}\right) = \dfrac{27}{50}$

Question 21

There are three boxes with the following compositions:

Colour Box	Blue	Red	White	Total
I	5	8	10	23
II	4	9	8	21
III	3	6	7	16

One ball in drawn from each box. What is the probability that they would be of the same colour?

Solution

All the three balls would either be Blue or Red or White. Denoting Blue, Red and White balls by *B*, *R* and *W* respectively and the box by lower suffix, the required probability is:

$$P(B_1 \cap B_2 \cap B_3) + P(R_1 \cap R_2 \cap R_3) + P(W_1 \cap W_2 \cap W_3)$$

$$= \left[P(B_1) \times P(B_2) \times P(B_3)\right] + \left[P(R_1) \times P(R_2) \times P(R_3)\right] + \left[P(W_1) \times P(W_2) \times P(W_3)\right]$$

$$= \left[\frac{5}{23} \times \frac{4}{21} \times \frac{3}{16}\right] + \left[\frac{8}{23} \times \frac{9}{21} \times \frac{6}{16}\right] + \left[\frac{10}{23} \times \frac{8}{21} \times \frac{7}{16}\right] = \frac{1052}{7728}$$

Random Variable – Probability Distribution

A random variable or **stochastic variable** is a function defined on a sample space associated with a random experiment. A random variable is denoted by a capital letter.

Consider this example: A coin is tossed three times, and we assign *X* to denote the number of heads. Here, *X* is known as the random variable. Here, the sample space is {(*HHH*), (*HHT*), (*HTH*), (*HTT*), (*THH*), (*THT*), (*TTH*), (*TTT*)}. Each element in the sample space is known as a sample point. For example, in the above sample space, "*HHH*" is a sample point; "*HHT*" is a sample point, and so on. Now, looking at each of the sample points, we can determine the value of our random variable *X*. In the sample point "*HHH*", the value of the random variable *X* is 3, as there are three heads; in the sample point "*HHT*", the value of the random variable *X* is 2, as there are two heads; in the sample point "*HTH*", the value of the random variable *X* is 2, as there are two heads; in the sample point "*HTT*", the value of the random variable *X* is 1, as there is one head; in the sample point "*TTT*", the value of the random variable *X* is 0, as there are no heads; and so on.

Based on the above discussion, we have:

Sample Point	*HHH*	*HHT*	*HTH*	*HTT*	*THH*	*THT*	*TTH*	*TTT*
Value of Random Variable *X*	3	2	2	1	2	1	1	0

Now, the probability that the value of random variable *X* will be 3 is 1/8 (as the probability of three heads is 1/8); the probability that the value of random variable *X* will be 2 is 3/8 (as the probability of two heads is 3/8), and so on. Therefore, the probability distribution of the random variable is given as follows:

X	0	1	2	3	Total
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

This tabular representation of the values of a random variable *X* and the corresponding probabilities is known as its probability distribution.

Expected Value of a Random Variable

Important Formulae

1. Expected value (μ) of a random variable (x) is given by: $\mu = E(x) = \sum p_i x_i$
2. Expected value of (x^2) is given by: $E(x^2) = \sum \left[p_i \left(x_i^2 \right) \right]$
3. Expected value of a monotonic function $\left[g(x) \right]$ is given by: $E\left[g(x) \right] = \sum \left[p_i \left\{ g(x) \right\} \right]$
4. Variance (σ^2) of a random variable (x) is given by: $V(x) = \sigma^2 = E(x-\mu)^2 = E(x^2) - \mu^2$
5. Standard Deviation (σ) of a random variable (x) is given by the positive square root of the variance.
6. If a and b are two constants related with two random variables x and y as $y = a + bx$, then the mean, i.e., the expected value of y is given by: $\mu_y = a + b\mu_x$.
7. If a and b are two constants related with two random variables x and y as $y = a + bx$, then the standard deviation of y is given by: $\sigma_y = |b| \times \sigma_x$.
8. If a and b are two constants related with two random variables x and y as $y = a + bx$, then the variance of y is given by: $(\sigma_y)^2 = (|b| \times \sigma_x)^2 = (b)^2 \times (\sigma_x)^2$.

Properties of Expected Value

1. Expectation of a constant *k* is *k*, i.e., $E(k) = k$, for any constant *k*.
2. Expectation of sum of two random variables is the sum of their expectations, i.e., $E(x+y) = E(x) + E(y)$, for any two random variables *x* and *y*.
3. Expectation of the product of a constant and a random variable is the product of the constant and the expectation of the random variable, i.e., $E(kx) = k.E(x)$, for any constant *k*.
4. Expectation of the product of two random variables is the product of the expectation of the two random variables, provided the two variables are independent, i.e., $E(x \times y) = E(x) \times E(y)$. This holds true whenever *x* and *y* are independent.

Question 22

An unbiased coin is tossed three times. Find the expected value of the number of heads and also its standard deviation.

Solution

Let the number of heads be denoted by the random variable *x*. Therefore, probability distribution of *x* is as follows:

x	0	1	2	3
p	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The expected value of *x* is given by:

$$\mu = E(x) = \sum p_i x_i = \left(\frac{1}{8}\times 0\right)+\left(\frac{3}{8}\times 1\right)+\left(\frac{3}{8}\times 2\right)+\left(\frac{1}{8}\times 3\right)=1.50$$

Variance of x is given by:

$V(x)=\sigma^2=E(x-\mu)^2=E(x^2)-\mu^2$, where, $E(x^2)=\sum\left[p_i\left(x_i^2\right)\right]$

Calculation of $E(x^2)$

p_i	x	x_i^2	$p_i x_i^2$
0.125	0.000	0.000	0.000
0.375	1.000	1.000	0.375
0.375	2.000	4.000	1.500
0.125	3.000	9.000	1.125
Total			$\sum\left[p_i\left(x_i^2\right)\right]=3.000$

Therefore, $V(x)=\sigma^2=E(x-\mu)^2=E(x^2)-\mu^2=3-(1.50)^2=0.75$

Standard Deviation $(\sigma)=\sqrt{0.75}=0.87$

Question 23

A random variable has the following probability distribution:

X	4	5	7	8	10
P	0.15	0.20	0.40	0.15	0.10

Find $E\left[x-E(x)\right]^2$. Also, obtain $v(3x-4)$.

Solution

To calculate $E\left[x-E(x)\right]^2$, we first need to calculate $E(x)=\sum p_i x_i$.

Therefore, $E(x)=\mu=\sum p_i x_i=(4\times 0.15)+(5\times 0.20)+(7\times 0.40)+(8\times 0.15)+(10\times 0.10)=6.60$

Now, $E\left[x-E(x)\right]^2$ is nothing but $E\left[x-\mu\right]^2$, which is the variance of x.

p_i	x_i	$x-\mu$	$[x-\mu]^2$	$p_i[x-\mu]^2$
0.150	4.000	–2.600	6.760	1.014
0.200	5.000	–1.600	2.560	0.512
0.400	7.000	0.400	0.160	0.064
0.150	8.000	1.400	1.960	0.294
0.100	10.000	3.400	11.560	1.156
Total				$\sum p_i[x-\mu]^2=3.040$

Therefore, variance of x, i.e., $E\left[x-E(x)\right]^2=E\left[x-\mu\right]^2=3.04$

Now, we need to calculate $v(3x-4)$. In this expression, "*v*" means the variance. Therefore, the requirement is to calculate the variance of $(3x-4)$. Let $3x-4$ be *y*.

Therefore, $y = 3x - 4 \Rightarrow y = -4 + 3x$

We know that if *a* and *b* are two constants related with two random variables *x* and *y* as $y = a + bx$, then the standard deviation of *y* is given by: $\sigma_y = |b| \times \sigma_x$. Therefore, variance of *y* is given by $(\sigma_y)^2 = (|b| \times \sigma_x)^2$.

In this equation, $y = -4 + 3x$, $a = -4$; $b = 3$

Therefore, $(\sigma_y)^2 = (|b| \times \sigma_x)^2 = (|3| \times \sqrt{3.04})^2 = (3)^2 \times (\sqrt{3.04})^2 = 9 \times 3.04 = 27.36$

Question 24

A random variable *x* has the following probability distribution:

x	0	1	2	3	4	5	6	7
$P(x)$	0	$2k$	$3k$	k	$2k$	k^2	$7k^2$	$2k^2 + k$

Find:

1. the value of *k*.
2. $P(x < 3)$
3. $P(x \geq 4)$
4. $P(2 < x \leq 5)$

Solution

1. We know that sum of probabilities is 1.
   Therefore, $0 + 2k + 3k + k + 2k + k^2 + 7k^2 + 2k^2 + k = 1$
   $\Rightarrow 10k^2 + 9k - 1 = 0 \Rightarrow (k+1)(10k-1) = 0 \Rightarrow k = -1; \, ^1\!/_{10}$
   Since *k* can't be negative, $k = \frac{1}{10} = 0.10$.
2. $P(x < 3) = P(0) + P(1) + P(2) = 0 + 2k + 3k = 5k = 5 \times 0.10 = 0.50$
3. $P(x \geq 4) = P(4) + P(5) + P(6) + P(7) = 2k + k^2 + 7k^2 + 2k^2 + k = 2(0.10) + (0.10)^2 + 7(0.10)^2 + 2(0.10)^2 + 0.10 = 0.40$.
4. $P(2 < x \leq 5) = P(3) + P(4) + P(5) = k + 2k + k^2 = 0.10 + 2(0.10) + (0.10)^2 = 0.31$