CHAPTER 5

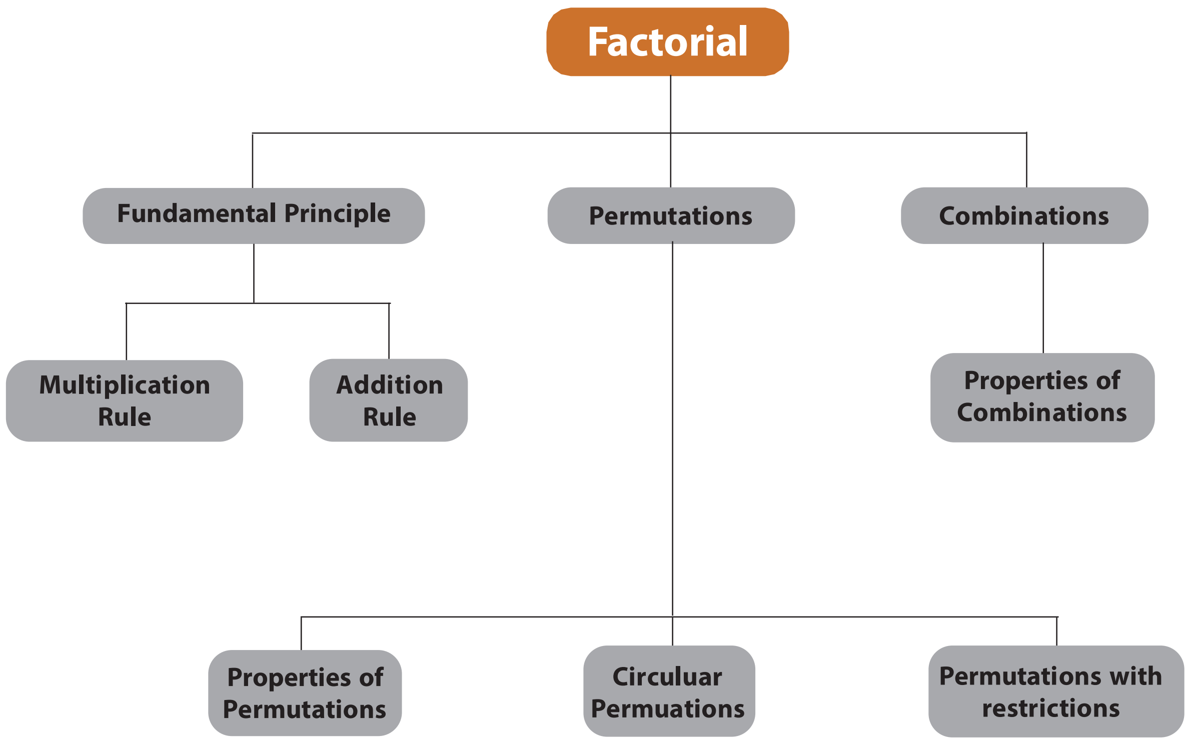
BASIC CONCEPTS OF PERMUTATIONS AND COMBINATIONS

LEARNING OBJECTIVES

After reading this Chapter a student will be able to understand —

- difference between permutation and combination for the purpose of arranging different objects;
- number of permutations and combinations when r objects are chosen out of n different objects.
- meaning and computational techniques of circular permutation and permutation with restrictions.

UNIT OVERVIEW



5.1 INTRODUCTION

In this chapter we will learn problem of arranging and grouping of certain things, taking particular number of things at a time. It should be noted that (a, b) and (b, a) are two different arrangements, but they represent the same group. In case of arrangements, the sequence or order of things is also taken into account.

The manager of a large bank has a difficult task of filling two important positions from a group of five equally qualified employees. Since none of them has had actual experience, he decides to allow each of them to work for one month in each of the positions before he makes the decision. How long can the bank operate before the positions are filled by permanent appointments?

Solution to above - cited situation requires an efficient counting of the possible ways in which the desired outcomes can be obtained. A listing of all possible outcomes may be desirable, but is likely to be very tedious and subject to errors of duplication or omission. We need to devise certain techniques which will help us to cope with such problems. The techniques of permutation and combination will help in tackling problems such as above.

FUNDAMENTAL PRINCIPLES OF COUNTING

(a) **Multiplication Rule:** If certain thing may be done in 'm' different ways and when it has been done, a second thing can be done in 'n ' different ways then total number of ways of doing both things simultaneously = m × n.

Eg. if one can go to school by 5 different buses and then come back by 4 different buses then total number of ways of going to and coming back from school = 5 × 4 = 20.

(b) **Addition Rule :** It there are two alternative jobs which can be done in 'm' ways and in 'n' ways respectively then either of two jobs can be done in (m + n) ways.

Eg. if one wants to go school by bus where there are 5 buses or to by auto where there are 4 autos, then total number of ways of going school = 5 + 4 = 9.

Note :- 1)

AND ⇒ Multiply
OR ⇒ Add

2) The above fundamental principles may be generalised, wherever necessary.

5.2 THE FACTORIAL

Definition: The factorial n, written as n! or $\underline{|n}$, represents the product of all integers from 1 to n both inclusive. To make the notation meaningful, when n = o, we define o! or $\underline{|o}$ = 1.

Thus, n! = n (n – 1) (n – 2) ….. …3.2.1

Example 1: Find 5!, 4! and 6!

Solution: 5! = 5 × 4 × 3 × 2 × 1 = 120; 4! = 4 × 3 × 2 × 1 = 24; 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720.

Example 2: Find 9 ! / 6 ! ; 10 ! / 7 !.

Solution: $\frac{9!}{6!} = \frac{9 \times 8 \times 7 \times 6!}{6!} = 9 \times 8 \times 7 = 504\,; \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!}$ = 10 × 9 × 8 =720

Example 3: Find x if 1/9 ! + 1/10 ! = x/11 !

Solution: 1/9! (1 + 1/10) = x/11 × 10 × 9! or, 11/10 = x/11 × 10 i.e., x = 121

Example 4: Find n if $\underline{|n+1} = 30\underline{|n-1}$

Solution: $\underline{|n+1} = 30\underline{|n-1} \Rightarrow (n+1).n\underline{|n-1} = 30\underline{|n-1}$

or, $n^2 + n = 30$ or, $n^2 + n - 30$ or, $n^2 + 6n - 5n - 30 = 0$ or, $(n + 6)(n - 5) = 0$

either n = 5 or n = –6. (Not possible) ∴ n = 5.

5.3 PERMUTATIONS

A group of persons want themselves to be photographed. They approach the photographer and request him to take as many different photographs as possible with persons standing in different positions amongst themselves. The photographer wants to calculate how many films does he need to exhaust all possibilities? How can he calculate the number?

In the situations such as above, we can use permutations to find out the exact number of films.

Definition: The ways of arranging or selecting smaller or equal number of persons or objects from a group of persons or collection of objects with due regard being paid to the order of arrangement or selection, are called permutations.

Let us explain, how the idea of permutation will help the photographer. Suppose the group consists of Mr. Suresh, Mr. Ramesh and Mr. Mahesh. Then how many films does the photographer need? He has to arrange three persons amongst three places with due regard to order. Then the various possibilities are (Suresh, Mahesh, Ramesh), (Suresh, Ramesh, Mahesh), (Ramesh, Suresh, Mahesh), (Ramesh, Mahesh, Suresh), (Mahesh, Ramesh, Suresh) and (Mahesh, Suresh, Ramesh). Thus there are six possibilities. Therefore he needs six films. Each one of these possibilities is called a permutation of three persons taken at a time.

This may also be exhibited as follows :

Alternative	Place 1	Place2	Place 3
1	Suresh……….	Mahesh………..	Ramesh
2	Suresh……….	Ramesh………..	Mahesh
3	Ramesh………	Suresh…………	Mahesh
4	Ramesh………	Mahesh………..	Suresh
5	Mahesh………	Ramesh………..	Suresh
6	Mahesh………	Suresh………….	Ramesh

with this example as a base, we can introduce a general formula to find the number of permutations.

Number of Permutations when r objects are chosen out of n different objects. (Denoted by ${}^nP_r$ or ${}_nP_r$ or $P_{(n,\,r)}$) :

Let us consider the problem of finding the number of ways in which the first *r* rankings are secured by n students in a class. As any one of the n students can secure the first rank, the number of ways in which the first rank is secured is n.

Now consider the second rank. There are (n – 1) students left and the second rank can be secured by any one of them. Thus the different possibilities are (n – 1) ways. Now, applying fundamental principle, we can see that the first two ranks can be secured in n (n – 1) ways by these n students.

After calculating for two ranks, we find that the third rank can be secured by any one of the remaining (n – 2) students. Thus, by applying the generalized fundamental principle, the first three ranks can be secured in n (n – 1) (n – 2) ways .

Continuing in this way we can visualise that the number of ways are reduced by one as the rank is increased by one. Therefore, again, by applying the generalised fundamental principle for r different rankings, we calculate the number of ways in which the first r ranks are secured by n students as

$${}^nP_r = n\,\{(n-1)\ldots\left(n-\overline{r-1}\right)\}$$

$$= n\,(n-1)\,\ldots\,(n-r+1)$$

Theorem : The number of permutations of n things chosen r at a time is given by

$${}^nP_r = n\,(n-1)\,(n-2)\ldots(n-r+1)$$

where the product has exactly r factors.

5.4 RESULTS

1 Number of permutations of n different things taken all n things at a time is given by

$${}^nP_n = n\,(n-1)\,(n-2)\,\ldots.\,(n-n+1)$$

$$= n\,(n-1)\,(n-2)\,\ldots..\,2.1 = n! \qquad \ldots(1)$$

2. ${}^nP_r$ using factorial notation.

$${}^nP_r = n.\,(n-1)\,(n-2)\,\ldots..\,(n-r+1)$$

$$= n\,(n-1)\,(n-2)\,\ldots..\,(n-r+1) \times \frac{(n-r)\,(n-r-1)\,2.1}{1.2\ldots(n-r-1)\,(n-r)}$$

$$= n!/(\,n-r\,)! \qquad \ldots(2)$$

Thus

$${}^nP_r = \frac{n!}{(n-r)!}$$

3. Justification for 0! = 1. Now applying r = n in the formula for ${}^nP_r$, we get

$^nP_n = n!/ (n - n)! = n!/0!$

But from Result 1 we find that $^nP_n = n!$. Therefore, by applying this we derive, $0! = n! / n! = 1$

Example 1: Evaluate each of $^5P_3$, $^{10}P_2$, $^{11}P_5$.

Solution: $^5P_3 = 5\times4\times (5-3+1) = 5 \times 4 \times 3 = 60$,

$^{10}P_2 = 10 \times \ldots \times (10-2+1) = 10 \times 9 = 90$,

$^{11}P_5 = 11! / (11 - 5)! = 11 \times 10 \times 9 \times 8 \times 7 \times 6! / 6! = 11 \times 10 \times 9 \times 8 \times 7 = 55440$.

Example 2: How many three letters words can be formed using the letters of the words (a) SQUARE and (b) HEXAGON?

(Any arrangement of letters is called a word even though it may or may not have any meaning or pronunciation).

Solution:

(a) Since the word 'SQUARE' consists of 6 different letters, the number of permutations of choosing 3 letters out of six equals $^6P_3 = 6 \times 5 \times 4 = 120$.

(b) Since the word 'HEXAGON' contains 7 different letters, the number of permutations is $^7P_3 = 7 \times 6 \times 5 = 210$.

Example 3: In how many different ways can five persons stand in a line for a group photograph?

Solution: Here we know that the order is important. Hence, this is the number of permutations of five things taken all at a time. Therefore, this equals

$^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Example 4: First, second and third prizes are to be awarded at an engineering fair in which 13 exhibits have been entered. In how many different ways can the prizes be awarded?

Solution: Here again, order of selection is important and repetitions are not meaningful as no exhibit can receive more than one prize. Hence , the answer is the number of permutations of 13 things taken three at a time. Therefore, we find $^{13}P_3 = 13!/10! = 13\times12\times11 = 1{,}716$ ways.

Example 5: In how many different ways can 3 students be associated with 4 chartered accountants, assuming that each chartered accountant can take at most one student?

Solution: This equals the number of permutations of choosing 3 persons out of 4. Hence , the answer is $^4P_3 = 4\times3\times2 = 24$.

Example 6: If six times the number permutations of n things taken 3 at a time is equal to seven times the number of permutations of (n – 1) things taken 3 at a time, find n.

Solution: We are given that $6 \times {}^nP_3 = 7 \times {}^{n-1}P_3$ and we have to solve this equality to find the value of n. Therefore,

$$6\frac{\lfloor n}{\lfloor n-3} = 7\frac{\lfloor n-1}{\lfloor n-4}$$

or, $6n(n - 1)(n - 2) = 7(n - 1)(n - 2)(n - 3)$

or, $6n = 7(n - 3)$

or, 6 n = 7n – 21

or, n = 21

Therefore, the value of n equals 21.

Example 7: Compute the sum of 4 digit numbers which can be formed with the four digits 1, 3, 5, 7, if each digit is used only once in each arrangement.

Solution: The number of arrangements of 4 different digits taken 4 at a time is given by $^4P_4 = 4! = 24$. All the four digits will occur equal number of times at each of the positions, namely ones, tens, hundreds, thousands.

Thus, each digit will occur 24 / 4 = 6 times in each of the positions. The sum of digits in one's position will be 6 × (1 + 3 + 5 + 7) = 96. Similar is the case in ten's, hundred's and thousand's places. Therefore, the sum will be 96 + 96 × 10 + 96 × 100 + 96 × 1000 = 1,06,656.

Example 8: Find n if $^nP_3 = 60$.

Solution: $^nP_3 = \frac{n!}{(n-3)!} = 60$ (given)

i.e., n (n–1) (n–2) = 60 = 5 × 4 × 3

Therefore, n = 5.

Example 9: If $^{56}P_{r+6} : {}^{54}P_{r+3} = 30{,}800 : 1$, find r.

Solution: We know $^np_r = \frac{n!}{(n-r)!}$;

$$\therefore {}^{56}P_{r+6} = \frac{56!}{\{56-(r+6)\}!} = \frac{56!}{(50-r)!}$$

$$\text{Similarly, } {}^{54}P_{r+3} = \frac{54!}{\{54-(r+3)\}!} = \frac{54!}{(51-r)!}$$

$$\text{Thus, } \frac{^{56}P_{r+6}}{^{54}P_{r+3}} = \frac{56!}{(50-r!)} \text{x} \frac{(51-r)!}{54!}$$

$$\frac{56\times55\times54!}{(50-r)!}\times\frac{(51-r)(50-r)!}{54!} = \frac{56\times55\times(51-r)}{1}$$

But we are given the ratio as 30800 : 1 ; therefore

$$\frac{56\times55\times(51-r)}{1} = \frac{30{,}800}{1}$$

$$\text{or, } (51-r) = \frac{30{,}800}{56\times55} = 10, \quad \therefore r = 41$$

Example 10: Prove the following

(n + 1)! – n! = ⇒ n.n!

Solution: By applying the simple properties of factorial, we have

(n +1)! – n! = (n+1) n! – n! = n!. (n+1–1) = n. n!

Example 11: In how many different ways can a club with 10 members select a President, Secretary and Treasurer, if no member can hold two offices and each member is eligible for any office?

Solution: The answer is the number of permutations of 10 persons chosen three at a time. Therefore, it is $^{10}p_3 = 10\times9\times8=720$.

Example 12: When Jhon arrives in New York, he has eight shops to see, but he has time only to visit six of them. In how many different ways can he arrange his schedule in New York?

Solution: He can arrange his schedule in $^{8}P_6 = 8\times 7 \times 6 \times 5 \times 4 \times 3 = 20{,}160$ ways.

Example 13: When Dr. Ram arrives in his dispensary, he finds 12 patients waiting to see him. If he can see only one patient at a time, find the number of ways, he can schedule his patients (a) if they all want their turn, and (b) if 3 leave in disgust before Dr. Ram gets around to seeing them.

Solution: (a) There are 12 patients and all 12 wait to see the doctor. Therefore the number of ways = $^{12}P_{12}$ = 12! = 479,001,600

(b) There are 12–3 = 9 patients. They can be seen $^{12}P_9$ = 79,833,600 ways.

EXERCISE 5 (A)

Choose the most appropriate option (a) (b) (c) or (d)

1. $^4P_3$ is evaluated as
   a) 43 b) 34 c) 24 d) None of these
2. $^4P_4$ is equal to
   a) 1 b) 24 c) 0 d) none of these
3. $\underline{|7}$ is equal to
   a) 5040 b) 4050 c) 5050 d) none of these
4. $\underline{|0}$ is a symbol equal to
   a) 0 b) 1 c) Infinity d) none of these
5. In ^nP_r, n is always
   a) an integer b) a fraction c) a positive integer d) none of these
6. In ^nP_r, the restriction is
   a) $n > r$ b) $n \geq r$ c) $n \leq r$ d) none of these
7. In $^nP_r = n\,(n-1)\,(n-2)\,\ldots\ldots\ldots\ldots(n-r+1)$, the number of factors is
   a) n b) r–1 c) n–r d) r
8. ^nP_r can also written as
   a) $\frac{\underline{|n}}{\underline{|n-r}}$ b) $\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ c) $\frac{\underline{|r}}{\underline{|n-r}}$ d) none of these

9 If $^nP_4 = 12 \times {}^nP_2$, the n is equal to

a) –1 b) 6 c) 5 d) none of these

10. If . $^nP_3$: $^nP_2$ = 3 : 1, then n is equal to
a) 7 b) 4 c) 5 d) none of these

11. $^{m+n}P_2$ = 56, $^{m-n}P_2$ = 30 then
a) m =6, n = 2 b) m = 7, n= 1 c) m=4,n=4 d) none of these

12. if ^5P_r = 60, then the value of r is
a) 3 b) 2 c) 4 d) none of these

13. If $^{n_1+n_2}P_2$ = 132, $^{n_1-n_2}P_2$ = 30 then,
a) n_1=6,n_2=6 b) n_1 = 10, n_2 = 2 c) n_1 = 9, n_2 = 3 d) none of these

14. The number of ways the letters of the word `COMPUTER' can be rearranged is
a) 40,320 b) 40,319 c) 40,318 d) none of these

15. The number of arrangements of the letters in the word `FAILURE', so that vowels are always coming together is
a) 576 b) 575 c) 570 d) none of these

16. 10 examination papers are arranged in such a way that the best and worst papers never come together. The number of arrangements is
a) $9\underline{|8}$ b) $\underline{|10}$ c) $8\underline{|9}$ d) none of these

17. n articles are arranged in such a way that 2 particular articles never come together. The number of such arrangements is
a) $(n-2)\ \underline{|n-1}$ b) $(n-1)\ \underline{|n-2}$ c) $\underline{|n}$ d) none of these

18. If 12 school teams are participating in a quiz contest, then the number of ways the first, second and third positions may be won is
a) 1,230 b) 1,320 c) 3,210 d) none of these

19. The sum of all 4 digit number containing the digits 2, 4, 6, 8, without repetitions is
a) 1,33,330 b) 1,22,220 c) 2,13,330 d) 1,33,320

20 The number of 4 digit numbers greater than 5,000 can be formed out of the digits 3,4,5,6 and 7(No. digit is repeated). The number of such is
a) 72 b) 27 c) 70 d) none of these

21. 4 digit numbers to be formed out of the figures 0, 1, 2, 3, 4 (no digit is repeated) then number of such numbers is
(a) 120 (b) 20 (c) 96. (d) none of these

22. The number of ways the letters of the word `TRIANGLE' to be arranged so that the word 'angle' will be always present is
(a) 20 (b) 60 (c) 24 (d) 32

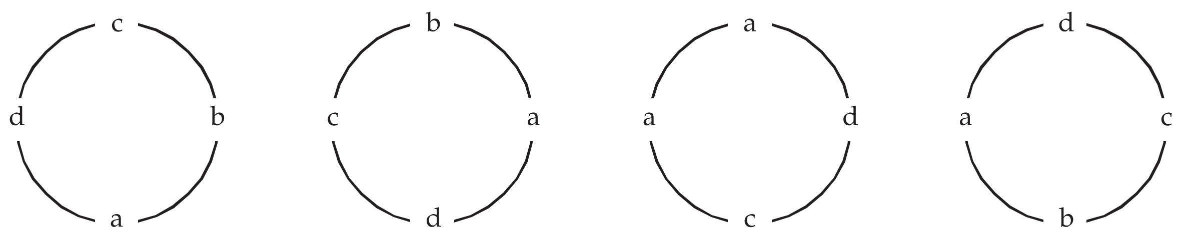
23. If the letters word 'DAUGHTER' are to be arranged so that vowels occupy the odd places, then number of different words are
(a) 2,880 (b) 676 (c) 625 (d) 576

5.5 CIRCULAR PERMUTATIONS

So for we have discussed arrangements of objects or things in a row which may be termed as linear permutation. But if we arrange the objects along a closed curve viz., a circle, the permutations are known as circular permutations.

The number of circular permutations of n different things chosen at a time is (n–1)!.

Proof : Let any one of the permutations of n different things taken. Then consider the rearrangement of this permutation by putting the last thing as the first thing. Eventhough this is a different permutation in the ordinary sense, it will not be different in all *n* things are arranged in a circle. Similarly, we can consider shifting the last two things to the front and so on. Specially, it can be better understood, if we consider *a,b,c,d*. If we place *a,b,c,d* in order, then we also get *abcd*, *dabc, cdab, bcda* as four ordinary permutations. These four words in circular case are one and same thing. See above circles.



abcd dabc cdab bcda

Thus we find in above illustration that four ordinary permutations equals one in circular.

Therefore, *n* ordinary permutations equal one circular permutation.

Hence there are ^nP_n/ n ways in which all the *n* things can be arranged in a circle. This equals (*n*–*1*)!.

Example 1: In how many ways can 4 persons sit at a round table for a group discussions?

Solution: The answer can be get from the formula for circular permutations. The answer is (4–1)! = 3! = 6 ways.

NOTE : These arrangements are such that every person has got the same two neighbours. The only change is that right side neighbour and vice-versa.

Thus the number of ways of arranging n persons along a round table so that no person has the same two neighbours is $= \frac{1}{2}\underline{|n-1}$

Similarly, in forming a necklace or a garland there is no distinction between a clockwise and anti clockwise direction because we can simply turn it over so that clockwise becomes anti clockwise and vice versa. **Hence, the number of necklaces formed with n beads of different colours**$= \frac{1}{2}\underline{|n-1}$

5.6 PERMUTATION WITH RESTRICTIONS

In many arrangements there may be number of restrictions. in such cases, we are to arrange or select the objects or persons as per the restrictions imposed. The total number of arrangements in all cases, can be found out by the application of fundamental principle.

Theorem 1. Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is $^{n-1}p_r$.

Proof : Since a particular object is always to be excluded, we have to place n – 1 objects at r places. Clearly this can be done in $^{n-1}p_r$ ways.

Theorem 2. Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is r. $^{n-1}p_{r-1}$

Proof : If the particular object is placed at first place, remaining r – 1 places can be filled from n – 1 distinct objects in $^{n-1}p_{r-1}$ ways. Similarly, by placing the particular object in 2nd, 3rd, ….., r^{th} place, we find that in each case the number of permutations is $^{n-1}p_{r-1}$. This the total number of arrangements in which a particular object always occurs is r. $^{n-1}p_{r-1}$

The following examples will enlighten further:

Example 1: How many arrangements can be made out of the letters of the word `DRAUGHT', the vowels never beings separated?

Solution: The word `DRAUGHT' consists of 7 letters of which 5 are consonants and two are vowels. In the arrangement we are to take all the 7 letters but the restriction is that the two vowels should not be separated.

We can view the two vowels as one letter. The two vowels A and U in this one letter can be arranged in 2! = 2 ways. (i) AU or (ii) UA. Further, we can arrange the six letters : 5 consonants and one letter compound letter consisting of two vowels. The total number of ways of arranging them is $^6P_6$ = 6! = 720 ways.

Hence, by the fundamental principle, the total number of arrangements of the letters of the word DRAUGHT, the vowels never being separated = 2 × 720 = 1440 ways.

Example 2: Show that the number of ways in which n books can be arranged on a shelf so that two particular books are not together. The number is $(n-2).(n-1)!$

Solution: We first find the total number of arrangements in which all n books can be arranged on the shelf without any restriction. The number is, $^nP_n = n!$ ….. (1)

Then we find the total number of arrangements in which the two particular books are together.

The books can be together in $^2P_2 = 2! = 2$ ways. Now we consider those two books which are kept together as one composite book and with the rest of the $(n-2)$ books from $(n-1)$ books which are to be arranged on the shelf; the number of arrangements = $^{n-1}P_{n-1} = (n-1)$!. Hence by the Fundamental Principle, the total number of arrangements on which the two particular books are together equals = $2 \times (n-1)!$ …….(2)

the required number of arrangements of n books on a shelf so that two particular books are not together

= (1) – (2)
= $n! - 2 \times (n-1)!$
= $n.(n - 1)! - 2 . (n-1)!$
= $(n-1)! . (n-2)$

Example 3: There are 6 books on Economics, 3 on Mathematics and 2 on Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?

Solution: Consider one such arrangement. 6 Economics books can be arranged among themselves in 6! Ways, 3 Mathematics books can be arranged in 3! Ways and the 2 books on Accountancy can be arranged in 2! ways. Consider the books on each subject as one unit. Now there are three units. These 3 units can be arranged in 3! Ways.

Total number of arrangements = $3! \times 6! \times 3! \times 2!$
= 51,840.

Example 4: How many different numbers can be formed by using any three out of five digits 1, 2, 3, 4, 5, no digit being repeated in any number?

How many of these will (i) begin with a specified digit? (ii) begin with a specified digit and end with another specified digit?

Solution: Here we have 5 different digits and we have to find out the number of permutations of them 3 at a time. Required number is ${}^5P_3 = 5.4.3 = 60$.

(i) If the numbers begin with a specified digit, then we have to find

the number of Permutations of the remaining 4 digits taken 2 at a time. Thus, desire number is ${}^4P_2 = 4.3 = 12$.

(ii) Here two digits are fixed; first and last; hence, we are left with the choice of finding the number of permutations of 3 things taken one at a time i.e., ${}^3P_1 =3$.

Example 5: How many four digit numbers can be formed out of the digits 1,2,3,5,7,8,9, if no digit is repeated in any number? How many of these will be greater than 3000?

Solution: We are given 7 different digits and a four-digit number is to be formed using any 4 of these digits. This is same as the permutations of 7 different things taken 4 at a time.

Hence, the number of four-digit numbers that can be formed = ${}^7P_4 = 7 \times 6 \times 5 \times 4 \times = 840$ ways.

Next, there is the restriction that the four-digit numbers so formed must be greater than 3,000. Thus, it will be so if the first digit-that in the thousand's position, is one of the five digits 3, 5, 7, 8, 9. Hence, the first digit can be chosen in 5 different ways; when this is done, the rest of the 3 digits are to be chosen from the rest of the 6 digits without any restriction and this can be done in ${}^6P_3$ ways.

Hence, by the Fundamental principle, we have the number of four-digit numbers greater than 3,000 that can be formed by taking 4 digits from the given 7 digits = $5 \times {}^6P_3 = 5 \times 6 \times 5 \times 4 = 5 \times 120 = 600$.

Example 6: Find the total number of numbers greater than 2000 that can be formed with the digits 1, 2, 3, 4, 5 no digit being repeated in any number.

Solution: All the 5 digit numbers that can be formed with the given 5 digits are greater than 2000. This can be done in

$^5P_5 = 5! = 120$ ways ....(1)

The four digited numbers that can be formed with any four of the given 5 digits are greater than 2000 if the first digit, i.e.,the digit in the thousand's position is one of the four digits 2, 3, 4, 5. this can be done in $^4P_1 = 4$ ways. When this is done, the rest of the 3 digits are to be chosen from the rest of 5–1 = 4 digits. This can be done in $^4P_3 = 4 \times 3 \times 2 = 24$ ways.

Therefore, by the Fundamental principle, the number of four-digit numbers greater than 2000

$= 4 \times 24 = 96$ …. (2)

Adding (1) and (2), we find the total number greater than 2000 to be 120 + 96 = 216.

Example 7: There are 6 students of whom 2 are Indians, 2 Americans, and the remaining 2 are Russians. They have to stand in a row for a photograph so that the two Indians are together, the two Americans are together and so also the two Russians. Find the number of ways in which they can do so.

Solution: The two Indians can stand together in $^2P_2 = 2! = 2$ ways. So is the case with the two Americans and the two Russians.

Now these 3 groups of 2 each can stand in a row in $^3P_3 = 3 \times 2 = 6$ ways. Hence by the generalized fundamental principle, the total number of ways in which they can stand for a photograph under given conditions is

$6 \times 2 \times 2 \times 2 = 48$

Example 8: A family of 4 brothers and three sisters is to be arranged for a photograph in one row. In how many ways can they be seated if (i) all the sisters sit together, (ii) no two sisters sit together?

Solution:

(i) Consider the sisters as one unit and each brother as one unit. 4 brothers and 3 sisters make 5 units which can be arranged in 5! ways. Again 3 sisters may be arranged amongst themselves in 3! Ways

Therefore, total number of ways in which all the sisters sit together = $5! \times 3! = 720$ ways.

(ii) In this case, each sister must sit on each side of the brothers. There are 5 such positions as indicated below by upward arrows :

↑ B1 ↑ B2 ↑ B3 ↑ B4 ↑

4 brothers may be arranged among themselves in 4! ways. For each of these arrangements 3 sisters can sit in the 5 places in $^5P_3$ ways.

Thus the total number of ways = $^5P_3 \times 4! = 60 \times 24 = 1{,}440$

Example 9: In how many ways can 8 persons be seated at a round table? In how many cases will 2 particular persons sit together?

Solution: This is in form of circular permutation. Hence the number of ways in which eight persons can be seated at a round table is $(n - 1)! = (8 - 1)! = 7! = 5040$ ways.

Consider the two particular persons as one person. Then the group of 8 persons becomes a group of 7 (with the restriction that the two particular persons be together) and seven persons can be arranged in a circular in 6! Ways.

Hence, by the fundamental principle, we have, the total number of cases in which 2 particular persons sit together in a circular arrangement of 8 persons = 2! × 6! = 2 × 6 × 5 × 4 × 3 ×2×1 = 1,440.

Example 10: Six boys and five girls are to be seated for a photograph in a row such that no two girls sit together and no two boys sit together. Find the number of ways in which this can be done.

Solution: Suppose that we have 11 chairs in a row and we want the 6 boys and 5 girls to be seated such that no two girls and no two boys are together. If we number the chairs from left to right, the arrangement will be possible if and only if boys occupy the odd places and girls occupy the even places in the row. The six odd places from 1 to 11 may filled in by 6 boys in ${}^6P_6$ ways. Similarly, the five even places from 2 to 10 may be filled in by 5 girls in ${}^5P_5$ ways.

Hence, by the fundamental principle, the total number of required arrangements = ${}^6P_6 \times {}^5P_5$ = 6! × 5! = 720 × 120 = 86,400.

EXERCISE 5 (B)

Choose the most appropriate option (a) (b) (c) or (d)

1 The number of ways in which 7 girls form a ring is
(a) 700 (b) 710 (c) 720 (d) none of these

2. The number of ways in which 7 boys sit in a round table so that two particular boys may sit together is
(a) 240 (b) 200 (c) 120 (d) none of these

3. If 50 different jewels can be set to form a necklace then the number of ways is
(a) $\frac{1}{2}\underline{|50}$ (b) $\frac{1}{2}\underline{|49}$ (c) $\underline{|49}$ (d) none of these

4. 3 ladies and 3 gents can be seated at a round table so that any two and only two of the ladies sit together. The number of ways is
(a) 70 (b) 27 (c) 72 (d) none of these

5. The number of ways in which the letters of the word `DOGMATIC' can be arranged is
(a) 40,319 (b) 40,320 (c) 40,321 (d) none of these

6. The number of arrangements of 10 different things taken 4 at a time in which one particular thing always occurs is
(a) 2015 (b) 2016 (c) 2014 (d) none of these

7. The number of permutations of 10 different things taken 4 at a time in which one particular thing never occurs is
(a) 3,020 (b) 3,025 (c) 3,024 (d) none of these

8. Mr. X and Mr. Y enter into a railway compartment having six vacant seats. The number of ways in which they can occupy the seats is
(a) 25 (b) 31 (c) 32 (d) 30

9. The number of numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6, 7 is
(a) 210 (b) 200 (c) 110 (d) none of these

10. The number of numbers lying between 10 and 1000 can be formed with the digits 2,3,4,0,8,9 is
(a) 124 (b) 120 (c) 125 (d) none of these

11. In a group of boys the number of arrangement of 4 boys is 12 times the number of arrangements of 2 boys. The number of boys in the group is
(a) 10 (b) 8 (c) 6 (d) none of these

12. The value of $\sum_{r=1}^{10} r.{}^{r}P_{r}$ is
(a) ${}^{11}P_{11}$ (b) ${}^{11}P_{11} - 1$ (c) ${}^{11}P_{11} + 1$ (d) none of these

13. The total number of 9 digit numbers of different digits is
(a) $10\underline{|9}$ (b) $8\underline{|9}$ (c) $9\underline{|9}$ (d) none of these

14. The number of ways in which 6 men can be arranged in a row so that the particular 3 men sit together, is
(a) ${}^{4}P_{4}$ (b) ${}^{4}P_{4} \times {}^{3}P_{3}$ (c) $(\underline{|3})^2$ (d) none of these

15. There are 5 speakers A, B, C, D and E. The number of ways in which A will speak always before B is
(a) 24 (b) $\underline{|4} \times \underline{|2}$ (c) $\underline{|5}$ (d) none of these

16. There are 10 trains plying between Calcutta and Delhi. The number of ways in which a person can go from Calcutta to Delhi and return by a different train is
(a) 99 (b) 90 (c) 80 (d) none of these

17. The number of ways in which 8 sweats of different sizes can be distributed among 8 persons of different ages so that the largest sweat always goes to be younger assuming that each one of then gets a sweat is
(a) $\underline{|8}$ (b) 5040 (c) 5039 (d) none of these

18. The number of arrangements in which the letters of the word `MONDAY' be arranged so that the words thus formed begin with M and do not end with N is
(a) 720 (b) 120 (c) 96 (d) none of these

19. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is
(a) $\underline{|7} / \underline{|3}$ (b) $\underline{|6} \times \underline{|7} / \underline{|3}$ (c) 35 (d) none of these

20. The number of ways in which the letters of the word `MOBILE' be arranged so that consonants always occupy the odd places is
(a) 36 (b) 63 (c) 30 (d) none of these.

21. 5 persons are sitting in a round table in such way that Tallest Person is always on the right–side of the shortest person; the number of such arrangements is
(a) 6 (b) 8 (c) 24 (d) none of these

5.7 COMBINATIONS

We have studied about permutations in the earlier section. There we have said that while arranging, we should pay due regard to order. There are situations in which order is not important. For example, consider selection of 5 clerks from 20 applicants. We will not be concerned about the order in which they are selected. In this situation, how to find the number of ways of selection? The idea of combination applies here.

Definition : The number of ways in which smaller or equal number of things are arranged or selected from a collection of things where the order of selection or arrangement is not important, are called combinations.

The selection of a poker hand which is a combination of five cards selected from 52 cards is an example of combination of 5 things out of 52 things.

Number of combinations of n different things taken r at a time. (denoted by ^nC_r C(n,r), $C_{n,r}$)

Let ^nC_r denote the required number of combinations. Consider any one of those combinations. It will contain r things. Here we are not paying attention to order of selection. Had we paid attention to this, we will have permutations or r items taken r at a time. In other words, every combination of r things will have ^rP_r permutations amongst them. Therefore, ^nC_r combinations will give rise to ^nC_r. ^rP_r permutations of r things selected from n things. From the earlier section, we can say that ^nC_r. ^rP_r = ^nP_r as ^nP_r denotes the number of permutations of r things chosen out of n things.

Since, $^nC_r . {^rP_r} = {^nP_r}$,

$^nC_r = {^nP_r}/{^rP_r}$ $= n!/(n-r)! \div r!/(r-r)!$

$= n!/(n-r)! \times 0!/r!$

$= n! / r!(n-r)!$

$\therefore {^nC_r} = n!/r!(n-r)!$

Remarks: Using the above formula, we get

(i) $^nC_o = n!/0!(n-0)! = n!/n! = 1$. [As 0! = 1]

$^nC_n = n!/n!(n-n)! = n!/n!0! = 1$ [Applying the formula for ^nC_r with r = n]

Example 1: Find the number of different poker hands in a pack of 52 playing cards.

Solution: This is the number of combinations of 52 cards taken five at a time. Now applying the formula,

$^{52}C_5 = 52!/5!\,(52-5)! = 52!/5!\,47! = \dfrac{52\times51\times50\times49\times48\times47!}{5\times4\times3\times2\times1\times47!}$

$= 2{,}598{,}960$

Example 2: Let S be the collection of eight points in the plane with no three points on the straight line. Find the number of triangles that have points of S as vertices.

Solution: Every choice of three points out of S determines a unique triangle. The order of the points selected is unimportant as whatever be the order, we will get the same triangle. Hence, the desired number is the number of combinations of eight things taken three at a time. Therefore, we get

$^8C_3 = 8!/3!5! = 8\times7\times6/3\times2\times1 = 56$ choices.

Example 3: A committee is to be formed of 3 persons out of 12. Find the number of ways of forming such a committee.

Solution: We want to find out the number of combinations of 12 things taken 3 at a time and this is given by

$^{12}C_3 = 12!/3!(12-3)!$ [by the definition of ^nC_r]

$= 12!/3!9! = 12\times11\times10\times9!/3!9! = 12\times11\times10/3\times2 = 220$

Example 4: A committee of 7 members is to be chosen from 6 Chartered Accountants, 4 Economists and 5 Cost Accountants. In how many ways can this be done if in the committee, there must be at least one member from each group and at least 3 Chartered Accountants?

Solution: The various methods of selecting the persons from the various groups are shown below:

Committee of 7 members			
	C.A.s	**Economists**	**Cost Accountants**
Method 1	3	2	2
Method 2	4	2	1
Method 3	4	1	2
Method 4	5	1	1
Method 5	3	3	1
Method 6	3	1	3

Number of ways of choosing the committee members by

Method 1 = $^6C_3\times{}^4C_2\times{}^5C_2 = \dfrac{6\times5\times4}{3\times2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5\times4}{2\times1}$ $=20\times6\times10=1{,}200$.

Method 2 = $^6C_4\times{}^4C_2\times{}^5C_1 = \dfrac{6\times5}{2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5}{1}$ $= 15\times6\times5 = 450$

Method 3 = $^6C_4\times{}^4C_1\times{}^5C_2 = \dfrac{6\times5}{2\times1}\times4\times\dfrac{5\times4}{2\times1}$ $= 15\times4\times10 = 600$.

Method 4 = $^6C_5 \times {}^4C_1 \times {}^5C_1$ = 6×4×5 = 120.

Method 5 = $^6C_3 \times {}^4C_3 \times {}^5C_1 = \frac{6\times5\times4}{3\times2\times1} \times \frac{4\times3\times2}{3\times2\times1} \times 5$ = 20×4×5 = 400.

Method 6 = $^6C_3 \times {}^4C_1 \times {}^5C_3 = \frac{6\times5\times4}{3\times2\times1} \times 4 \times \frac{5\times4}{2\times1}$ = 20×4×10 = 800.

Therefore, total number of ways = 1,200 + 450 + 600 + 120 + 400 + 800 = 3,570

Example 5: A person has 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests such that 5 of them are relatives?

Solution: Of the 12 friends, 8 are relatives and the remaining 4 are not relatives. He has to invite 5 relatives and 2 friends as his guests. 5 relatives can be chosen out of 8 in $^8C_5$ ways; 2 friends can be chosen out of 4 in $^4C_2$ ways.

Hence, by the fundamental principle, the number of ways in which he can invite 7 guests such that 5 of them are relatives and 2 are friends.

$= {}^8C_5 \times {}^4C_2$

$= \{8! / 5!\,(8-5)!\} \times \{4! / 2!\,(4-2)!\} = \left[(8 \times 7 \times 6 \times 5!)/5! \times 3!\right] \times \frac{4\times3\times2\times!}{2!\,2!} = 8\times7\times6$

$= 336.$

Example 6: A Company wishes to simultaneously promote two of its 6 department heads to assistant managers. In how many ways these promotions can take place?

Solution: This is a problem of combination. Hence, the promotions can be done in

$^6C_2 = 6\times5 / 2 = 15$ ways

Example 7: A building contractor needs three helpers and ten men apply. In how many ways can these selections take place?

Solution: There is no regard for order in this problem. Hence, the contractor can select in any of $^{10}C_3$ ways i.e.,

$(10 \times 9 \times 8) / (3 \times 2 \times 1) = 120$ ways.

Example 8: In each case, find n:

Solution: (a) $4.\,{}^nC_2 = {}^{n+2}C_3$ (b) $^{n+2}C_n = 45$.

(a) We are given that $4.\,{}^nC_2 = {}^{n+2}C_3$. Now applying the formula,

$$4\times\frac{n!}{2!(n-2)!} = \frac{(n+2)!}{3!(n+2-3)!}$$

or, $$\frac{4\times n.(n-1)(n-2)!}{2!(n-2)!} = \frac{(n+2)\,(n+1)\,.\,n\,.\,(n-1)!}{3!\,(n-1)!}$$

$$4n(n-1) /2 = (n+2)\,(n+1)n /3!$$

or, $4n(n-1) / 2 = (n+2)(n+1)n / 3\times2\times1$

or, $12(n-1)=(n+2)(n+1)$

or, $12n-12 = n^2 + 3n + 2$

or, $n^2 - 9n + 14 = 0$.

or, $n^2 - 2n - 7n + 14 = 0$.

or, $(n-2)(n-7) = 0$

$\therefore$ $n=2$ or 7.

(b) We are given that ${}^{n+2}C_n = 45$. Applying the formula,

$(n+2)!/\{n!(n+2-n)!\} = 45$

or, $(n+2)(n+1)\, n! / n!\, 2! = 45$

or, $(n+1)(n+2) = 45\times2! = 90$

or, $n^2+3n-88 = 0$

or, $n^2+11n-8n-88 = 0$

or, $(n+11)(n-8) = 0$

Thus, n equals either – 11 or 8. But negative value is not possible. Therefore we conclude that n=8.

Example 9: A box contains 7 red, 6 white and 4 blue balls. How many selections of three balls can be made so that (a) all three are red (b) none is red (c) one is of each colour?

Solution: (a) All three balls will be of red colour if they are taken out of 7 red balls and this can be done in

${}^{7}C_3 = 7! / 3!(7-3)!$

$= 7! / 3!4! = 7\times6\times5\times4! / (3\times2\times4!) = 7\times6\times5 / (3\times2) = 35$ ways

Hence, 35 selections (groups) will be there such that all three balls are red.

(b) None of the three will be red if these are chosen from (6 white and 4 blue balls) 10 balls and this can be done in

${}^{10}C_3 = 10!/\{3!(10-3)!\} = 10! / 3!7!$

$= 10\times9\times8\times7! / (3\times2\times1\times7!) = 10\times9\times8 / (3\times2) = 120$ ways.

Hence, the selections (or groups) of three such that none is a red ball are 120 in number.

One red ball can be chosen from 7 balls in ${}^{7}C_1 = 7$ ways. One white ball can be chosen from 6 white balls in ${}^{6}C_1$ ways. One blue ball can be chosen from 4 blue balls in ${}^{4}C_1 = 4$ ways. Hence, by generalized fundamental principle, the number of groups of three balls such that one is of each colour = $7\times6\times4 = 168$ ways.

Example 10: If ${}^{10}P_r = 6{,}04{,}800$ and ${}^{10}C_r = 120$; find the value of r,

Solution: We know that ${}^{n}C_r \cdot {}^{r}P_r = {}^{n}P_r$. We will us this equality to find r.

${}^{10}P_r = {}^{10}C_r \cdot r!$

or, $6{,}04{,}800 = 120 \times r!$

or, $r! = 6{,}04{,}800 \div 120 = 5{,}040$

But $r! = 5040 = 7\times6\times4\times3\times2\times1 = 7!$

Therefore, r=7.

Properties of ^nC_r :

1. $^nC_r = {^nC_{n-r}}$

   We have $^nC_r = n! / \{r!(n-r)!\}$ and $^nC_{n-r} = n! / [(n-r)!\,\{n-(n-r)\}!] = n! / \{(n-r)!(n-n+r)!\}$

   Thus $^nC_{n-r} = n! / \{(n-r)!\,(n-n+r)!\} = n! / \{(n-r)!r!\} = {^nC_r}$

2. $^{n+1}C_r = {^nC_r} + {^nC_{r-1}}$

   By definition,

   $^nC_{r-1} + {^nC_r} = n! / \{(r-1)!\,(n-r+1)!\} + n! / \{r!(n-r)!\}$

   But $r! = r\times(r-1)!$ and $(n-r+1)! = (n-r+1) \times (n-r)!$. Substituting these in above, we get

$$^nC_{r-1} + {^nC_r} = n!\left\{\frac{1}{(r-1)!(n-r+1)(n-r)!} + \frac{1}{r(r-1)!\,(n-r)!}\right\}$$

$$= \{n! / (r-1)!\,(n-r)!\}\ \{(1 / n-r+1) + (1/r)\}$$

$$= \{n! / (r-1)!\,(n-r)!\}\ \{(r+n-r+1) / r(n-r+1)\}$$

$$= (n+1)\,n! / \{r\,.\,(r-1)!\,(n-r)!\,.\,(n-r+1)\}$$

$$= (n+1)! / \{r!(n+1-r)!\} = {^{n+1}C_r}$$

3. $\mathbf{^nC_o} = n!/\{0!\,(n-0)!\} = n! / n! = \mathbf{1}$.
4. $\mathbf{^nC_n} = n!/\{n!\,(n-n)!\} = n! / n!\,.\,0! = \mathbf{1}$.

Note

(a) ^nC_r has a meaning only when r and n are integers $0\le r \le n$ and $^nC_{n-r}$ has a meaning only when $0 \le n - r \le n$.

(b) ^nC_r and $^nC_{n-r}$ are called complementary combinations, for if we form a group of r things out of n different things, (n–r) remaining things which are not included in this group form another group of rejected things. The number of groups of n different things, taken r at a time should be equal to the number of groups of n different things taken (n–r) at a time.

Example 11: Find r if $^{18}C_r = {^{18}C_{r+2}}$

Solution: As $^nC_r = {^nC_{n-r}}$, we have $^{18}C_r = {^{18}C_{18-r}}$

But it is given, $^{18}C_r = {^{18}C_{r+2}}$

$\therefore\ ^{18}C_{18-r} = {^{18}C_{r+2}}$

or, $18 - r = r+2$

Solving, we get

$2r = 18 - 2 = 16$ i.e., $r=8$.

Example 12: Prove that

${}^{n}C_{r} = {}^{n-2}C_{r-2} + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r}$

Solution: R.H.S $= {}^{n-2}C_{r-2} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r}$

$= {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ [using Property 2 listed earlier]

$= {}^{(n-1)+1}C_{r}$ [using Property 2 again]

$= {}^{n}C_{r}$ = L.H.S.

Hence, the result

Example 13: If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.

Solution: We have ${}^{n}C_{r} = n! / \{r!(n-r)!\}$ Now, substituting for n and r, we get

${}^{28}C_{2r} = 28! / \{(2r)!(28 - 2r)!\}$,

${}^{24}C_{2r-4} = 24! / [(2r-4)!\{24 - (2r-4)\}!] = 24! / \{(2r-4)!(28-2r)!\}$

We are given that ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$. Now we calculate,

$$\frac{{}^{28}C_{2r}}{{}^{24}C_{2r-4}} = \frac{28!}{(2r)!(28-2r)!} \div \frac{(2r-4)!(28-2r)!}{24!}$$

$$= \frac{28\times27\times26\times25\times24!}{(2r)(2r-1)(2r-2)(2r-3)(2r-4)!(28-2r)!} \times \frac{(2r-4)!(28-2r)!}{24!}$$

$$= \frac{28\times27\times26\times25}{(2r)(2r-1)(2r-2)(2r-3)} = \frac{225}{11}$$

$$\text{or, } (2r)(2r-1)(2r-2)(2r-3) = \frac{11\times28\times27\times26\times25}{225}$$

$= 11\times28\times3\times26$

$= 11\times7\times4\times3\times13\times2$

$= 11\times12\times13\times14$

$= 14\times13\times12\times11$

$\therefore$ $2r = 14$ i.e., $r = 7$

Example 14: Find x if ${}^{12}C_{5} + 2\,{}^{12}C_{4} + {}^{12}C_{3} = {}^{14}C_{x}$

Solution: L.H.S $= {}^{12}C_{5} + 2\,{}^{12}C_{4} + {}^{12}C_{3}$

$= {}^{12}C_{5} + {}^{12}C_{4} + {}^{12}C_{4} + {}^{12}C_{3}$

$= {}^{13}C_{5} + {}^{13}C_{4}$

$= {}^{14}C_{5}$

Also ${}^{n}C_{r} = {}^{n}C_{n-r}$. Therefore ${}^{14}C_{5} = {}^{14}C_{14-5} = {}^{14}C_{9}$

Hence, L.H.S = ${}^{14}C_5 = {}^{14}C_9 = {}^{14}C_x$ = R.H.S by the given equality

This implies, either x = 5 or x = 9.

Example 15 : Prove by reasoning that

(i) ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$

(ii) ${}^{n}P_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

Solution: (i) ${}^{n+1}C_r$ stands for the number of combinations of (n+1) things taken r at a time. As a specified thing can either be included in any combination or excluded from it, the total number of combinations which can be combinations or (n+1) things taken r at a time is the sum of :

(a) combinations of (n+1) things taken r at time in which one specified thing is always included and

(b) the number of combinations of (n+1) things taken r at time from which the specified thing is always excluded.

Now, in case (a), when a specified thing is always included , we have to find the number of ways of selecting the remaining (r–1) things out of the remaining n things which is ${}^{n}C_{r-1}$.

Again, in case (b), since that specified thing is always excluded, we have to find the number of ways of selecting r things out of the remaining n things, which is ${}^{n}C_r$.

Thus, ${}^{n+1}C_r = {}^{n}C_{r-1} + {}^{n}C_r$

(i) We divide ${}^{n}P_r$ i.e., the number of permutations of n things take r at a time into two groups:

(a) those which contain a specified thing

(b) those which do not contain a specified thing.

In (a) we fix the particular thing in any one of the r places which can be done in r ways and then fill up the remaining (r–1) places out of (n–1) things which give rise to ${}^{n-1}P_{r-1}$ ways. Thus, the number of permutations in case (a) = $r \times {}^{n-1}P_{r-1}$.

In case (b), one thing is to be excluded; therefore, r places are to be filled out of (n–1) things. Therefore, number of permutations = ${}^{n-1}P_r$

Thus, total number of permutations = ${}^{n-1}P_r + r.\,{}^{n-1}P_{r-1}$

i.e., ${}^{n}P_r = {}^{n-1}P_r + r.\,{}^{n-1}P_{r-1}$

5.8 STANDARD RESULTS

We now proceed to examine some standard results in permutations and combinations. These results have special application and hence are dealt with separately.

I. Permutations when some of the things are alike, taken all at a time

The number of ways *p* in which n things may be arranged among themselves, taking them all at a time, when n_1 of the things are exactly alike of one kind , n_2 of the things are exactly alike of another kind, n_3 of the things are exactly alike of the third kind, and the rest all are different is given by,

$$p = \frac{n!}{n_1!n_2!n_3!}$$

Proof : Let there be n things. Suppose n_1 of them are exactly alike of one kind; n_2 of them are exactly alike of another kind; n_3 of them are exactly alike of a third kind; let the rest $(n-n_1-n_2-n_3)$ be all different.

Let p be the required permutations; then if the n things, all exactly alike of one kind were replaced by n, different things different from any of the rest in any of the p permutations without altering the position of any of the remaining things, we could form $n_1!$ new permutations. Hence, we should obtain $p \times n_1!$ permutations.

Similarly if n_2 things exactly alike of another kind were replaced by n_2 different things different form any of the rest, the number of permutations would be $p \times n_1! \times n_2!$

Similarly, if n_3 things exactly alike of a third kind were replaced by n_3 different things different from any of the rest, the number of permutations would be $p \times n_1! \times n_2! \times n_3! = n!$

But now because of these changes all the n things are different and therefore, the possible number of permutations when all of them are taken is n!.

Hence, $p \times n_1! \times n_2!\, n_3! = n!$

i.e., $p = \dfrac{n!}{n_1!n_2!n_3!}$

which is the required number of permutations. This results may be extended to cases where there are different number of groups of alike things.

II. Permutations when each thing may be repeated once, twice,...upto r times in any arrangement.

Result: The number of permutations of n things taken r at time when each thing may be repeated r times in any arrangement is n^r.

Proof: There are n different things and any of these may be chosen as the first thing. Hence, there are n ways of choosing the first thing.

When this is done, we are again left with n different things and any of these may be chosen as the second (as the same thing can be chosen again.)

Hence, by the fundamental principle, the two things can be chosen in $n \times n = n^2$ number of ways.

Proceeding in this manner, and noting that at each stage we are to chose a thing from n different things, the total number of ways in which r things can be chosen is obviously equal to $n \times n \times$ ....to r terms $= n^r$.

III. Combinations of n different things taking some or all of n things at a time

Result : The total number of ways in which it is possible to form groups by taking some or all of n things $(2^n - 1)$.

In symbols, $\sum_{r=1}^{n} {}^{n}C_r = 2^n - 1$

Proof : Each of the n different things may be dealt with in two ways; it may either be taken or left. Hence, by the generalised fundamental principle, the total number of ways of dealing with n things :

$2 \times 2 \times 2 \times \ldots\ldots.2$, n times i.e., 2^n

But this include the case in which all the things are left, and therefore, rejecting this case, the total number of ways of forming a group by taking some or all of n different things is $2^n - 1$.

IV. Combinations of n things taken some or all at a time when n_1 of the things are alike of one kind, n_2 of the things are alike of another kind n_3 of the things are alike of a third kind. etc.

Result : The total, number of ways in which it is possible to make groups by taking some or all out of n $(=n_1 + n_2 + n_3 + \ldots)$ things, where n_1 things are alike of one kind and so on, is given by

$\{ (n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots\} -1$

Proof : The n_1 things all alike of one kind may be dealt with in $(n_1 + 1)$ ways. We may take 0, 1, 2,….n, of them. Similarly n_2 things all alike of a second kind may be dealt with in $(n_2 +1)$ ways and n_3 things all alike of a third kind may de dealt with in $(n_3 +1)$ ways.

Proceeding in this way and using the generalised fundamental principle, the total number of ways of dealing with all n $(= n_1 + n_2 + n_3 +\ldots)$ things, where n_1, things are alike of one kind and so on, is given by

$(n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots$

But this includes the case in which none of the things are taken. Hence, rejecting this case, total number of ways is $\{(n_1 + 1) (n_2 + 1) (n_3 + 1)\ldots\} -1\}$

V. The notion of Independence in Combinations

Many applications of Combinations involve the selection of subsets from two or more independent sets of objects or things. If the combination of a subset having r_1 objects form a set having n_1 objects does not affect the combination of a subset having r_2 objects from a different set having n_2 objects, we call the combinations as being independent. Whenever such combinations are independent, any subset of the first set of objects can be combined with each subset of the second set of the object to form a bigger combination. The total number of such combinations can be found by applying the generalised fundamental principle.

Result : The combinations of selecting r_1 things from a set having n_1 objects and r_2 things from a set having n_2 objects where combination of r_1 things, r_2 things are independent is given by

$$ {}^{n_1}C_{r_1} \times {}^{n_2}C_{r_2} $$

Note : This result can be extended to more than two sets of objects by a similar reasoning.

Example 1: How many different permutations are possible from the letters of the word `CALCULUS'?

Solution: The word `CALCULUS' consists of 8 letters of which 2 are C and 2 are L, 2 are U and the rest are A and S. Hence , by result (I), the number of different permutations from the letters of the word `CALCULUS' taken all at a time

$$= \frac{8!}{2!2!2!1!1!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 2 \times 2} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5{,}040$$

Example 2: In how many ways can 17 billiard balls be arranged , if 7 of them are black, 6 red and 4 white?

Solution: We have, the required number of different arrangements:

$$\frac{17!}{7!\,6!\,4!} = 40{,}84{,}080$$

Example 3: An examination paper with 10 questions consists of 6 questions in Algebra and 4 questions in Geometry. At least one question from each section is to be attempted. In how many ways can this be done?

Solution: A student must answer atleast one question from each section and he may answer all questions from each section.

Consider Section I : Algebra. There are 6 questions and he may answer a question or may not answer it. These are the two alternatives associated with each of the six questions. Hence, by the generalised fundamental principle, he can deal with two questions in 2 × 2 ….6 factors = 2^6 number of ways. But this includes the possibility of none of the question from Algebra being attempted. This cannot be so, as he must attempt at least one question from this section. Hence, excluding this case, the number of ways in which Section I can be dealt with is $(2^6 - 1)$.

Similarly, the number of ways in which Section II can be dealt with is $(2^4 - 1)$.

Hence, by the Fundamental Principle, the examination paper can be attempted in $(2^6 - 1)(2^4 - 1)$ number of ways.

Example 4: A man has 5 friends. In how many ways can he invite one or more of his friends to dinner?

Solution: By result, (III) of this section, as he has to select one or more of his 5 friends, he can do so in $2^5 - 1 = 31$ ways.

Note : This can also be done in the way, outlines below. He can invite his friends one by one, in twos, in threes, etc. and hence the number of ways.

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= 5 + 10 + 10 + 5 + 1 = 31 \text{ ways.}$$

Example 5: There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 can be formed of them if the committee is to include atleast two ladies?

Solution: The committee of six must include at least 2 ladies, i.e., two or more ladies. As there are only 3 ladies, the following possibilities arise:

The committee of 6 consists of (i) 4 men and 2 ladies (ii) 3 men and 3 ladies.

The number of ways for (i) = $^7C_4 \times {^3C_2} = 35 \times 3 = 105$;

The number of ways for (ii) = $^7C_3 \times {^3C_3} = 35 \times 1 = 35$.

Hence the total number of ways of forming a committee so as to include at least two ladies = 105 +35 = 140.

Example 6: Find the number of ways of selecting 4 letters from the word `EXAMINATION'.

Solution: There are 11 letters in the word of which A, I, N are repeated twice.

Thus we have 11 letters of 8 different kinds (A, A), (I, I), (N, N), E, X, M, T, O.

The group of four selected letters may take any of the following forms:

(i) Two alike and other two alike

(ii) Two alike and other two different

(iii) All four different

In case (i), the number of ways = $^3C_2 = 3$.

In case (ii), the number of ways = $^3C_1 \times {^7C_2} = 3 \times 21 = 63$.

In case (iii), the number of ways = $^8C_4 = \dfrac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$

Hence , the required number of ways = 3 + 63 + 70 = 136 ways

SUMMARY

- **Fundamental principles of counting**
  - (a) **Multiplication Rule:** If certain thing may be done in 'm' different ways and when it has been done, a second thing can be done in 'n ' different ways then total number of ways of doing both things simultaneously = m × n.
  - (b) **Addition Rule :** It there are two alternative jobs which can be done in 'm' ways and in 'n' ways respectively then either of two jobs can be done in (m + n) ways.
- **Factorial:** The factorial n, written as n! or $\underline{|n}$, represents the product of all integers from 1 to n both inclusive. To make the notation meaningful, when n = o, we define o! or $\underline{|o}$ = 1.

  Thus, n! = n (n – 1) (n – 2) ….. …3.2.1
- **Permutations:** The ways of arranging or selecting smaller or equal number of persons or objects from a group of persons or collection of objects with due regard being paid to the order of arrangement or selection, are called permutations.

The number of permutations of n things chosen r at a time is given by

$$^nP_r = n\ (n-1)(n-2)\ldots(n-r+1)$$

where the product has exactly r factors.

- **Circular Permutations:** (a) n ordinary permutations equal one circular permutation.

  Hence there are $^nP_n/\ n$ ways in which all the n things can be arranged in a circle. This equals $(n-1)!$.

  (b) the number of necklaces formed with n beads of different colours = $= \frac{1}{2}\underline{|n-1}$.

- (a) Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is $^{n-1}p_r$.

  (b) Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is r. $^{n-1}p_{r-1}$

- **Combinations:** The number of ways in which smaller or equal number of things are arranged or selected from a collection of things where the order of selection or arrangement is not important, are called combinations.

  $^nC_r = n!/r!\,(n-r)!$

  $^nC_r = {}^nC_{n-r}$

  $\mathbf{^nC_o} = n!/\{0!\ (n-0)!\} = n!\ /\ n! = \mathbf{1}$.

  $\mathbf{^nC_n} = n!/\{n!\ (n-n)!\} = n!\ /\ n!\ .\ 0! = \mathbf{1}$.

(a) ^nC_r has a meaning only when r and n are integers $0 \le r \le n$ and $^nC_{n-r}$ has a meaning only when $0 \le n - r \le n$.

(i) $^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(ii) $^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

- Permutations when some of the things are alike, taken all at a time

$$p = \frac{n!}{n_1!n_2!n_3!}$$

- Permutations when each thing may be repeated once, twice,…upto r times in any arrangement = n!.

- The total number of ways in which it is possible to form groups by taking some or all of n things $(2^n - 1)$.

- The total, number of ways in which it is possible to make groups by taking some or all out of n $(=n_1 + n_2 + n_3 + \ldots)$ things, where n_1 things are alike of one kind and so on, is given by

  $\{(n_1 + 1)(n_2 + 1)(n_3 + 1)\ldots\} - 1$

- The combinations of selecting r_1 things from a set having n_1 objects and r_2 things from a set having n_2 objects where combination of r_1 things, r_2 things are independent is given by

  $^{n_1}C_{r_1} \times {}^{n_2}C_{r_2}$

EXERCISE 5 (C)

Choose the most appropriate option (a, b, c or d)

1. The value of $^{12}C_4 + {}^{12}C_3$ is
   (a) 715 (b) 710 (C) 716 (d) none of these
2. If $^np_r = 336$ and $^nC_r = 56$, then n and r will be
   (a) (3, 2) (b) (8, 3) (c) (7, 4) (d) none of these
3. If $^{18}C_r = {}^{18}C_{r+2}$, the value of $^rC_5$ is
   (a) 55 (b) 50 (c) 56 (d) none of these
4. If $^n c_{r-1} = 56$, $^nc_r = 28$ and $^n c_{r+1} = 8$, then r is equal to
   (a) 8 (b) 6 (c) 5 (d) none of these
5. A person has 8 friends. The number of ways in which he may invite one or more of them to a dinner is.
   (a) 250 (b) 255 (c) 200 (d) none of these
6. The number of ways in which a person can chose one or more of the four electrical appliances : T.V, Refrigerator, Washing Machine and a cooler is
   (a) 15 (b) 25 (c) 24 (d) none of these
7. If $^nc_{10} = {}^nc_{14}$, then $^{25}c_n$ is
   (a) 24 (b) 25 (c) 1 (d) none of these
8. Out of 7 gents and 4 ladies a committee of 5 is to be formed. The number of committees such that each committee includes at least one lady is
   (a) 400 (b) 440 (c) 441 (d) none of these
9. If $^{28}c_{2r} : {}^{24}c_{2r-4} = 225 : 11$, then the value of r is
   (a) 7 (b) 5 (c) 6 (d) none of these
10. The number of diagonals in a decagon is
    (a) 30 (b) 35 (c) 45 (d) none of these
    Hint: The number of diagonals in a polygon of n sides is $\frac{1}{2}n(n-3)$.
11. There are 12 points in a plane of which 5 are collinear. The number of triangles is
    (a) 200 (b) 211 (c) 210 (d) none of these
12. The number of straight lines obtained by joining 16 points on a plane, no three of them being on the same line is
    (a) 120 (b) 110 (c) 210 (d) none of these

13. At an election there are 5 candidates and 3 members are to be elected. A voter is entitled to vote for any number of candidates not greater than the number to be elected. The number of ways a voter choose to vote is

(a) 20 (b) 22 (c) 25 (d) none of these

14. Every two persons shakes hands with each other in a party and the total number of hand shakes is 66. The number of guests in the party is
(a) 11 (b) 12 (c) 13 (d) 14

15. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 18 (c) 12 (d) 9

16. The number of ways in which 12 students can be equally divided into three groups is
(a) 5775 (b) 7575 (c) 7755 (d) none of these

17. The number of ways in which 15 mangoes can be equally divided among 3 students is

(a) $\lfloor 15 / \lfloor (5)^4$ (b) $\lfloor 15 / \lfloor (5)^3$ (c) $\lfloor 15 / \lfloor (5)^2$ (d) none of these

18. 8 points are marked on the circumference of a circle. The number of chords obtained by joining these in pairs is
(a) 25 (b) 27 (c) 28 (d) none of these

19. A committee of 3 ladies and 4 gents is to be formed out of 8 ladies and 7 gents. Mrs. X refuses to serve in a committee in which Mr. Y is a member. The number of such committees is
(a) 1530 (b) 1500 (c) 1520 (d) 1540

20. If ${}^{500}C_{92} = {}^{499}C_{92} + {}^{n}C_{91}$ then n is

(a) 501 (b) 500 (c) 502 (d) 499

21. The Supreme Court has given a 6 to 3 decision upholding a lower court; the number of ways it can give a majority decision reversing the lower court is
(a) 256 (b) 276 (c) 245 (d) 226.

22. Five bulbs of which three are defective are to be tried in two bulb points in a dark room. Number of trials the room shall be lighted is
(a) 6 (b) 8 (c) 5 (d) 7.

MISCELLANEOUS EXAMPLE

EXERCISE 5(D)

Choose the appropriate option a,b,c or d

1. The letters of the words `CALCUTTA' and `AMERICA' are arranged in all possible ways. The ratio of the number of there arrangements is

(a) 1:2 (b) 2:1 (c) 2:2 (d) none of these

2. The ways of selecting 4 letters from the word `EXAMINATION' is

(a) 136 (b) 130 (c) 125 (d) none of these

3. The number of different words that can be formed with 12 consonants and 5 vowels by taking 4 consonants and 3 vowels in each word is

(a) ${}^{12}c_4 \times {}^{5}c_3$ (b) ${}^{17}c_7$ (c) $4950 \times \underline{|7}!$ (d) none of these

4. Eight guests have to be seated 4 on each side of a long rectangular table.2 particular guests desire to sit on one side of the table and 3 on the other side. The number of ways in which the sitting arrangements can be made is

(a) 1732 (b) 1728 (c) 1730 (d) 1278.

5 A question paper contains 6 questions, each having an alternative.

The number of ways an examine can answer one or more questions is

(a) 720 (b) 728 (c) 729 (d) none of these

6. ${}^{51}c_{31}$ is equal to

(a) ${}^{51}c_{20}$ (b) $2.{}^{50}c_{20}$ (c) $2.{}^{45}c_{15}$ (d) none of these

7. The number of words that can be made by rearranging the letters of the word APURNA so that vowels and consonants appear alternate is

(a) 18 (b) 35 (c) 36 (d) none of these

8. The number of arrangement of the letters of the word `COMMERCE' is

(a) $\underline{|8}$ (b) $\underline{|8} / (\underline{|2}\underline{|2}\underline{|2})$ (c) 7 ! (d) none of these

9. A candidate is required to answer 6 out of 12 questions which are divided into two groups containing 6 questions in each group. He is not permitted to attempt not more than four from any group. The number of choices are.

(a) 750 (b) 850 (c) 800 (d) none of these

10. The results of 8 matches (Win, Loss or Draw) are to be predicted. The number of different forecasts containing exactly 6 correct results is

(a) 316 (b) 214 (c) 112 (d) none of these

11. The number of ways in which 8 different beads be strung on a necklace is

(a) 2500 (b) 2520 (c) 2250 (d) none of these

12. The number of different factors the number 75,600 has is

(a) 120 (b) 121 (c) 119 (d) none of these

13. The number of 4 digit numbers formed with the digits 1, 1, 2, 2, 3, 4 is

(a) 100 (b) 101 (c) 201 (d) none of these

14. The number of ways a person can contribute to a fund out of 1 ten-rupee note, 1 five-rupee note, 1 two-rupee and 1 one rupee note is

(a) 15 (b) 25 (c) 10 (d) none of these

15. The number of ways in which 9 things can be divided into twice groups containing 2,3, and 4 things respectively is

(a) 1250 (b) 1260 (c) 1200 (d) none of these

16. $^{(n-1)}P_r + r.^{(n-1)}P_{(r-1)}$ is equal to

(a) ^nC_r (b) $\underline{|n}/(\underline{|r}\ \underline{|n-r})$ (c) ^np_r (d) none of these

17. $\underline{|2n}$ can be written as

(a) $2^n\{1.3.5\ldots.(2n-1)\}\underline{|n}$ (b) $2^n\underline{|n}$ (c) $\{1.3.5\ldots..(2n-1)\}$ (d) none of these

18. The number of even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5 without repetion is

(a) 110 (b) 112 (c) 111 (d) none of these

19. 5 letters are written and there are five letter-boxes. The number of ways the letters can be dropped into the boxes, are in each

(a) 119 (b) 120 (c) 121 (d) none of these

20. $^nC_1 + {^nC_2} + {^nC_3} + {^nC_4} + \ldots..+ {^nC_n}$ equals

(a) 2^n-1 (b) 2^n (c) 2^n+1 (d) none of these

ANSWERS

Exercise 5(A)

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(b)	**5.**	(c)	**6.**	(b)	**7.**	(d)	**8.**	(a)
9.	(b)	**10.**	(c)	**11.**	(b)	**12.**	(b)	**13.**	(c)	**14.**	(b)	**15.**	(a)	**16.**	(c)
17.	(a)	**18.**	(b)	**19.**	(d)	**20.**	(a)	**21**	(c)	**22**	(c)	**23**	(a)		

Exercise 5 (B)

1.	(c)	**2.**	(a)	**3.**	(b)	**4.**	(c)	**5.**	(b)	**6.**	(b)	**7.**	(c)	**8.**	(d)
9.	(a)	**10.**	(c)	**11.**	(c)	**12.**	(b)	**13.**	(c)	**14.**	(b)	**15.**	(a)	**16.**	(b)
17.	(b)	**18.**	(c)	**19.**	(c)	**20.**	(a)	**21**	(a)						

Exercise 5 (C)

1.	(a)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(b)	**8.**	(c)
9.	(a)	**10.**	(b)	**11.**	(c)	**12.**	(a)	**13.**	(c)	**14.**	(b)	**15.**	(b)	**16.**	(a)
17.	(b)	**18.**	(c)	**19.**	(d)	**20.**	(d)	**21.**	(a)	**22.**	(d)				

Exercise 5 (D)

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(c)	**8.**	(b)&(c)
9.	(b)	**10.**	(c)	**11.**	(b)	**12.**	(c)	**13.**	(d)	**14.**	(a)	**15.**	(b)	**16.**	(c)
17.	(a)	**18.**	(c)	**19.**	(b)	**20.**	(a)								

ADDITIONAL QUESTION BANK

1. There are 6 routes for journey from station A to station B. In how many ways you may go from A to B and return if for returning you make a choice of any of the routes?

   (a) 6 (b) 12 (c) 36 (d) 30

2. As per question No.(1) if you decided to take the same route you may do it in _______ number of ways.

   (a) 6 (b) 12 (c) 36 (d) 30

3. As per question No.(1) if you decided not to take the same route you may do it in _______ number of ways.

   (a) 6 (b) 12 (c) 36 (d) 30

4. How many telephones connections may be allotted with 8 digits form the numbers 0,1,2 …….9?

   (a) 10^8 (b) 10! (c) ${}^{10}C_8$ (d) ${}^{10}P_8$

5. In how many different ways 3 rings of a lock can not combine when each ring has digits 0,1,2……9 leading to unsuccessful events?

   (a) 999 (b) 10^3 (c) 10! (d) 997

6. A dealer provides you Maruti Car & Van in 2 body patterns and 5 different colours. How many choices are open to you?

   (a) 2 (b) 7 (c) 20 (d) 10

7. 3 persons go into a railway carriage having 8 seats. In how many ways they may occupy the seats?

   (a) ${}^8P_3$ (b) ${}^8C_3$ (c) ${}^8C_5$ (d) None

8. Find how many five-letter words can be formed out of the word "LOGARITHMS" (the words may not convey any meaning)

   (a) ${}^{10}P_5$ (b) ${}^{10}C_5$ (c) ${}^9C_4$ (d) None

9. How many 4 digits numbers greater than 7000 can be formed out of the digits 3,5,7,8,9?

   (a) 24 (b) 48 (c) 72 (d) 50

10. In how many ways 5 Sanskrit 3 English and 3 Hindi books be arranged keeping the books of the same language together?

   (a) 5! × 3! × 3! × 3! (b) 5! × 3! × 3! (c) ${}^5P_3$ (d) None

11. In how many ways can 6 boys and 6 girls be seated around a table so that no 2 boys are adjacent?

(a) $4! \times 5!$ (b) $5! \times 6!$ (c) ${}^6P_6$ (d) $5 \times {}^6P_6$

12. In how many ways can 4 Americans and 4 English men be seated at a round table so that no 2 Americans may be together?

(a) $4! \times 3!$ (b) ${}^4P_4$ (c) $3 \times {}^4P_4$ (d) ${}^4C_4$

13. The chief ministers of 17 states meet to discuss the hike in oil price at a round table. In how many ways they seat themselves if the Kerala and Bengal chief ministers choose to sit together?

(a) $15! \times 2!$ (b) $17! \times 2!$ (c) $16! \times 2!$ (d) None

14. The number of permutation of the word `ACCOUNTANT' is

(a) $10! \div (2!)^4$ (b) $10! \div (2!)^3$ (c) $10!$ (d) None

15. The number of permutation of the word `ENGINEERING' is

(a) $11! \div [(3!)^2(2!)^2]$ (b) $11!$ (c) $11! \div [(3!)(2!)]$ (d) None

16. The number of arrangements that can be made with the word `ASSASSINATION' is

(a) $13! \div [3! \times 4! \times (2!)^2]$ (b) $13! \div [3! \times 4! \times 2!]$ (c) $13!$ (d) None

17. How many numbers higher than a million can be formed with the digits 0,4,4,5,5,5,3?

(a) 420 (b) 360 (c) $7!$ (d) None

18. The number of permutation of the word `ALLAHABAD' is

(a) $9! \div (4! \times 2!)$ (b) $9! \div 4!$ (c) $9!$ (d) None

19. In how many ways the vowels of the word `ALLAHABAD' will occupy the even places?

(a) 120 (b) 60 (c) 30 (d) None

20. How many arrangements can be made with the letter of the word `MATHEMATICS'?

(a) $11! \div (2!)^3$ (b) $11! \div (2!)^2$ (c) $11!$ (d) None

21. In how many ways of the word `MATHEMATICS' be arranged so that the vowels occur together?

(a) $11! \div (2!)^3$ (b) $(8! \times 4!) \div (2!)^3$ (c) $12! \div (2!)^3$ (d) None

22. In how many ways can the letters of the word `ARRANGE' be arranged?

(a) 1,200 (b) 1,250 (c) 1,260 (d) 1,300

23. In how many ways the word `ARRANGE' be arranged such that the 2 'R's come together?

(a) 400 (b) 440 (c) 360 (d) None

24. In how many ways the word `ARRANGE' be arranged such that the 2 'R's do not come together?

(a) 1,000 (b) 900 (c) 800 (d) None

25. In how many ways the word `ARRANGE' be arranged such that the 2 'R's and 2 'A's come together?

(a) 120 (b) 130 (c) 140 (d) None

26. If $^{n}P_{4} = 12,\ ^{n}P_{2}$ the value of n is

(a) 12 (b) 6 (c) -1 (d) both 6 -1

27. If $4.\,^{n}P_{3} = 5.\,^{n-1}P_{3}$ the value of n is

(a) 12 (b) 13 (c) 14 (d) 15

28. $^{n}P_{r} \div \,^{n-1}P_{r-1}$ is

(a) n (b) $n!$ (c) $(n-1)!$ (d) $^{n}C_{n}$

29. The total number of numbers less than 1000 and divisible by 5 formed with 0,1,2,…..9 such that each digit does not occur more than once in each number is

(a) 150 (b) 152 (c) 154 (d) None

30. The number of ways in which 8 examination papers be arranged so that the best and worst papers never come together is

(a) $8! - 2 \times 7!$ (b) $8! - 7!$ (c) $8!$ (d) None

31. In how many ways can 4 boys and 3 girls stand in a row so that no two girls are together?

(a) $5! \times 4! \div 3!$ (b) $^{5}P_{3} \times 3$ (c) $^{5}P_{3} \times 2$ (d) None

32. In how many ways can 3 boys and 4 girls be arranged in a row so that all the three boys are together?

(a) $4! \times 3!$ (b) $5! \times 3!$ (c) $7!$ (d) None

33. How many six digit numbers can be formed out of 4 5 …..9 no digits being repeated?

(a) $6! - 5!$ (b) $6!$ (c) $6! + 5!$ (d) None

34. In terms of question No.(33) how many of them are not divisible by 5?

(a) 6! – 5! (b) 6! (c) 6! + 5! (d) None

35. In how many ways the word `FAILURE' can be arranged so that the consonants occupy only the odd positions?

(a) 4! (b) $(4!)^2$ (c) 7! ÷ 3! (d) None

36. In how many ways can the word `STRANGE' be arranged so that the vowels are never separated?

(a) 6! × 2! (b) 7! (c) 7! ÷ 2! (d) None

37. In how many ways can the word `STRANGE' be arranged so that the vowels never come together?

(a) 7! – 6! × 2! (b) 7! – 6! (c) $^7P_6$ (d) None

38. In how many ways can the word `STRANGE' be arranged so that the vowels ocupy only the odd places?

(a) $^5P_5$ (b) $^5P_5 \times {}^4P_4$ (c) $^5P_5 \times {}^4P_2$ (d) None

39. How many four digits number can be formed by using 1,2, ……..7?

(a) $^7P_4$ (b) $^7P_3$ (c) $^7C_4$ (d) None

40. How many four digits numbers can be formed by using 1,2, …..7 which are grater than 3400?

(a) 500 (b) 550 (c) 560 (d) None

41. In how many ways it is possible to write the word `ZENITH' in a dictionary?

(a) $^6P_6$ (b) $^6C_6$ (c) $^6P_0$ (d) None

42. In terms of question No.(41) what is the rank or order of the word `ZENITH' in the dictionary?

(a) 613 (b) 615 (c) 616 (d) 618

43. If $^{n-1}P_3 \div {}^{n+1}P_3 = \frac{5}{12}$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

44. If $^{n+3}P_6 \div {}^{n+2}P_4 = 14$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

45. If $^7P_n \div {}^7P_{n-3} = 60$ the value of n is

(a) 8 (b) 4 (c) 5 (d) 2

46. There are 4 routes for going from Dumdum to Sealdah and 5 routes for going from Sealdah to Chandni. In how many different ways can you go from Dumdum to Chandni via Sealdah?

(a) 9 (b) 1 (c) 20 (d) None

47. In how many ways can 5 people occupy 8 vacant chairs?

(a) 5,720 (b) 6,720 (c) 7,720 (d) None

48. If there are 50 stations on a railway line how many different kinds of single first class tickets may be printed to enable a passenger to travel from one station to other?

(a) 2,500 (b) 2,450 (c) 2,400 (d) None

49. How many six digits numbers can be formed with the digits 9, 5, 3, 1, 7, 0?

(a) 600 (b) 720 (c) 120 (d) None

50. In terms of question No.(49) how many numbers will have 0's in ten's place?

(a) 600 (b) 720 (c) 120 (d) None

51. How many words can be formed with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

52. How many words can be formed beginning with 'N' with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

53. How many words can be formed beginning with 'N' and ending in 'A' with the letters of the word `SUNDAY'?

(a) 6! (b) 5! (c) 4! (d) None

54. How many different arrangements can be made with the letters of the word `MONDAY'?

(a) 6! (b) 8! (c) 4! (d) None

55. How many different arrangements can be made with the letters of the word `ORIENTAL'?

(a) 6! (b) 8! (c) 4! (d) None

56. How many different arrangements can be made beginning with 'A' and ending in 'N' with the letters of the word `MONDAY'?

(a) 6! (b) 8! (c) 4! (d) None

57. How many different arrangements can be made beginning with 'A' and ending with 'N' with the letters of the word `ORIENTAL'?

(a) 6! (b) 8! (c) 4! (d) None

58. In how many ways can a consonant and a vowel be chosen out of the letters of the word `LOGARITHM'?

(a) 18 (b) 15 (c) 3 (d) None

59. In how many ways can a consonant and a vowel be chosen out of the letters of the word `EQUATION'?

(a) 18 (b) 15 (c) 3 (d) None

60. How many different words can be formed with the letters of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) 2! × 6!

61. How many different words can be formed beginning with 'T' of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) 2! × 6!

62. How many different words can be formed beginning with 'E' of the letters of the word `TRIANGLE'?

(a) 8! (b) 7! (c) 6! (d) 2! × 6!

63. In question No. (60) how many of them will begin with 'T' and end with 'E'?

(a) 8! (b) 7! (c) 6! (d) 2! × 6!

64. In question No.(60) how many of them have 'T' and 'E' in the end places?

(a) 8! (b) 7! (c) 6! (d) 2! × 6!

65. In question No.(60) how many of them have consonants never together?

(a) 8! – 4! × 5! (b) ${}^6P_3 \times 5!$ (c) 2! × 5!×3! (d) ${}^4P_3 \times 5!$

66. In question No.(60) how many of them have arrangements that no two vowels are together?

(a) 8! – 4! × 5! (b) ${}^6P_3 \times 5!$ (c) 2! × 5! ×3! (d) ${}^4P_3 \times 5!$

67. In question No.(60) how many of them have arrangements that consonants and vowels are always together?

(a) 8! – 4! × 5! (b) ${}^6P_3 \times 5!$ (c) 2! × 5! ×3! (d) ${}^4P_3 \times 5!$

68. In question No.(60) how many of them have arrangements that vowels occupy odd places?

(a) 8! – 4! × 5! (b) $^6P_3 \times 5!$ (c) 2! × 5! ×3! (d) $^4P_3 \times 5!$

69. In question No.(60) how many of them have arrangements that the relative positions of the vowels and consonants remain unchanged?

(a) 8! – 4! × 5! (b) $^6P_3 \times 5!$ (c) 2! × 5! ×3! (d) 5! × 3!

70. In how many ways the letters of the word `FAILURE' can be arranged with the condition that the four vowels are always together?

(a) $(4!)^2$ (b) 4! (c) 7! (d) None

71. In how many ways n books can be arranged so that two particular books are not together?

(a) $(n - 2) \times (n - 1)!$ (b) $n \times n!$ (c) $(n - 2) \times (n - 2)!$ (d) None

72. In how many ways can 3 books on Mathematics and 5 books on English be placed so that books on the same subject always remain together?

(a) 1,440 (b) 240 (c) 480 (d) 144

73. 6 papers are set in an examination out of which two are mathematical. In how many ways can the papers be arranged so that 2 mathematical papers are together?

(a) 1,440 (b) 240 (c) 480 (d) 144

74. In question No.(73) will your answer be different if 2 mathematical papers are not consecutive?

(a) 1,440 (b) 240 (c) 480 (d) 144

75. The number of ways the letters of the word `SIGNAL' can be arranged such that the vowels occupy only odd positions is________.

(a) 1,440 (b) 240 (c) 480 (d) 144

76. In how many ways can be letters of the word `VIOLENT' be arranged so that the vowels occupy even places only?

(a) 1,440 (b) 240 (c) 480 (d) 144

77. How many numbers between 1000 and 10000 can be formed with 1, 2, …..9?

(a) 3,024 (b) 60 (c) 78 (d) None

78. How many numbers between 3000 and 4000 can be formed with 1, 2, …..6?

(a) 3,024 (b) 60 (c) 78 (d) None

79. How many numbers greater than 23,000 can be formed with 1, 2, …..5?

(a) 3,024 (b) 60 (c) 78 (d) None

80. If you have 5 copies of one book, 4 copies of each of two books, 6 copies each of three books and single copy of 8 books you may arrange it in ________number of ways.

(a) $\frac{39!}{5!\times(4!)^2\times(6!)^3}$ (b) $\frac{39!}{5!\times8!\times(4!)^2\times(6!)^3}$ (c) $\frac{39!}{5!\times8!\times4!\times(6!)^2}$ (d) $\frac{39!}{5!\times8!\times4!\times6!}$

81. How many arrangements can be made out of the letters of the word "PERMUTATION"?

(a) $\frac{1}{2}\,{}^{11}P_{11}$ (b) ${}^{11}P_{11}$ (c) ${}^{11}C_{11}$ (d) None

82. How many numbers greater than a million can be formed with the digits: One 0 Two 1 One 3 and Three 7?

(a) 360 (b) 240 (c) 840 (d) 20

83. How many arrangements can be made out of the letters of the word `INTERFERENCE' so that no two consonant are together?

(a) 360 (b) 240 (c) 840 (d) 20

84. How many different words can be formed with the letter of the word "HARYANA"?

(a) 360 (b) 240 (c) 840 (d) 20

85. In question No.(84) how many arrangements are possible keeping 'H' and 'N' together?

(a) 360 (b) 240 (c) 840 (d) 20

86. In question No.(84) how many arrangements are possible beginning with 'H' and ending with 'N'?

(a) 360 (b) 240 (c) 840 (d) 20

87. A computer has 5 terminals and each terminal is capable of four distinct positions including the positions of rest what is the total number of signals that can be made?

(a) 20 (b) 1,020 (c) 1,023 (d) None

88. In how many ways can 9 letters be posted in 4 letter boxes?

(a) 4^9 (b) 4^5 (c) ${}^9P_4$ (d) ${}^9C_4$

89. In how many ways can 8 beads of different colour be strung on a ring?

(a) 7! ÷ 2 (b) 7! (c) 8! (d) 8! ÷ 2

90. In how many ways can 8 boys form a ring?

(a) 7! ÷ 2 (b) 7! (c) 8! (d) 8! ÷ 2

91. In how many ways 6 men can sit at a round table so that all shall not have the same neighbours in any two occasions?

(a) 5! ÷ 2 (b) 5! (c) $(7!)^2$ (d) 7!

92. In how many ways 6 men and 6 women sit at a round table so that no two men are together?

(a) 5! ÷ 2 (b) 5! (c) $(5!)^2$ (d) 7!

93. In how many ways 4 men and 3 women are arranged at a round table if the women never sit together?

(a) 6 × 6! (b) 6! (c) 7! (d) None

94. In how many ways 4 men and 3 women are arranged at a round table if the women always sit together?

(a) 6 × 6! (b) 6! (c) 7! (d) None

95. A family comprised of an old man, 6 adults and 4 children is to be seated is a row with the condition that the children would occupy both the ends and never occupy either side of the old man. How many sitting arrangements are possible?

(a) 4! × 5! × 7! (b) 4! × 5! × 6! (c) 2! × 4! × 5! × 6! (d) None

96. The total number of sitting arrangements of 7 persons in a row if 3 persons sit together in a particular order is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

97. The total number of sitting arrangements of 7 persons in a row if 3 persons sit together in any order is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

98. The total number of sitting arrangements of 7 persons in a row if two persons occupy the end seats is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

99. The total number of sitting arrangements of 7 persons in a row if one person occupies the middle seat is _________.

(a) 5! (b) 6! (c) 2! × 5! (d) None

100. If all the permutations of the letters of the word `CHALK' are written in a dictionary the rank of this word will be ____________.

(a) 30 (b) 31 (c) 32 (d) None

101. In a ration shop queue 2 boys, 2 girls, and 2 men are standing in such a way that the boys the girls and the men are together each. The total number of ways of arranging the queue is ______.

(a) 42 (b) 48 (c) 24 (d) None

102. If you have to make a choice of 7 questions out of 10 questions set, you can do it in _______ number of ways.

(a) ${}^{10}C_7$ (b) ${}^{10}P_7$ (c) $7! \times {}^{10}C_7$ (d) None

103. From 6 boys and 4 girls 5 are to be seated. If there must be exactly 2 girls the number of ways of selection is ______.

(a) 240 (b) 120 (c) 60 (d) None

104. In your office 4 posts have fallen vacant. In how many ways a selection out of 31 candidates can be made if one candidate is always included?

(a) ${}^{30}C_3$ (b) ${}^{30}C_4$ (c) ${}^{31}C_3$ (d) ${}^{31}C_4$

105. In question No.(104) would your answer be different if one candidate is always excluded?

(a) ${}^{30}C_3$ (b) ${}^{30}C_4$ (c) ${}^{31}C_3$ (d) ${}^{31}C_4$

106. Out of 8 different balls taken three at a time without taking the same three together more than once for how many number of times you can select a particular ball?

(a) ${}^{7}C_2$ (b) ${}^{8}C_3$ (c) ${}^{7}P_2$ (d) ${}^{8}P_3$

107. In question No.(106) for how many number of times you can select any ball?

(a) ${}^{7}C_2$ (b) ${}^{8}C_3$ (c) ${}^{7}P_2$ (d) ${}^{8}P_3$

108. In your college Union Election you have to choose candidates. Out of 5 candidates 3 are to be elected and you are entitled to vote for any number of candidates but not exceeding the number to be elected. You can do it in _________ ways.

(a) 25 (b) 5 (c) 10 (d) None

109. In a paper from 2 groups of 5 questions each you have to answer any 6 questions attempting at least 2 questions from each group. This is possible in ________ number of ways.

(a) 50 (b) 100 (c) 200 (d) None

110. Out of 10 consonants and 4 vowels how many words can be formed each containing 6 consonant and 3 vowels?

(a) $^{10}C_6 \times {}^4C_3$ (b) $^{10}C_6 \times {}^4C_3 \times 9!$ (c) $^{10}C_6 \times {}^4C_3 \times 10!$ (d) None

111. A boat's crew consist of 8 men, 3 of whom can row only on one side and 2 only on the other. The number of ways in which the crew can be arranged is _________.

(a) $^3C_1 \times (4!)^2$ (b) $^3C_1 \times 4!$ (c) $^3C_1$ (d) None

112. A party of 6 is to be formed from 10 men and 7 women so as to include 3 men and 3 women. In how many ways the party can be formed if two particular women refuse to join it?

(a) 4,200 (b) 600 (c) 3,600 (d) None

113. You are selecting a cricket team of first 11 players out of 16 including 4 bowlers and 2 wicket-keepers. In how many ways you can do it so that the team contains exactly 3 bowlers and 1 wicket-keeper?

(a) 960 (b) 840 (c) 420 (d) 252

114. In question No.(113) would your answer be different if the team contains at least 3 bowlers and at least 1 wicket-keeper?

(a) 2,472 (b) 960 (c) 840 (d) 420

115. A team of 12 men is to be formed out of *n* persons. Then the number of times 2 men 'A' and 'B' are together is ___________.

(a) $^nC_{12}$ (b) $^{n-1}C_{11}$ (c) $^{n-2}C_{10}$ (d) None

116. In question No.(115) the number of times 3 men 'C' 'D' and 'E' are together is _____.

(a) $^nC_{12}$ (b) $^{n-1}C_{11}$ (c) $^{n-2}C_{10}$ (d) $^{n-2}C_{10}$

117. In question No.(115) it is found that 'A' and 'B' are three times as often together as 'C' 'D' and 'E' are. Then the value of *n* is ____________.

(a) 32 (b) 23 (c) 9 (d) None

118. The number of combinations that can be made by taking 4 letters of the word `COMBINATION' is _______.

(a) 70 (b) 63 (c) 3 (d) 136

119. If $^{18}C_n = {}^{18}C_{n+2}$ then the value of *n* is __________

(a) 0 (b) –2 (c) 8 (d) None

120. If $^{n}C_6 \div {}^{n-2}C_3 = \frac{91}{4}$ then the value of n is __________

(a) 15 (b) 14 (c) 13 (d) None

121. In order to pass PE-II examination minimum marks have to be secured in each of 7 subjects. In how many ways can a pupil fail?

(a) 128 (b) 64 (c) 127 (d) 63

122. In how many ways you can answer one or more questions out of 6 questions each having an alternative?

(a) 728 (b) 729 (c) 128 (d) 129

123. There are 12 points in a plane no 3 of which are collinear except that 6 points which are collinear. The number of different straight lines is _________.

(a) 50 (b) 51 (c) 52 (d) None

124. In question No.(123) the number of different triangles formed by joining the straight lines is ________.

(a) 220 (b) 20 (c) 200 (d) None

125. A committee is to be formed of 2 teachers and 3 students out of 10 teachers and 20 students. The numbers of ways in which this can be done is ______.

(a) $^{10}C_2 \times {}^{20}C_3$ (b) $^{9}C_1 \times {}^{20}C_3$ (c) $^{10}C_2 \times {}^{19}C_3$ (d) None

126. In question No.(125) if a particular teacher is included the number of ways in which this can be done is _________.

(a) $^{10}C_2 \times {}^{20}C_3$ (b) $^{9}C_1 \times {}^{20}C_3$ (c) $^{10}C_2 \times {}^{19}C_3$ (d) None

127. In question No.(125) if a particular student is excluded the number of ways in which this can be done is _________.

(a) $^{10}C_2 \times {}^{20}C_3$ (b) $^{9}C_1 \times {}^{20}C_3$ (c) $^{10}C_2 \times {}^{19}C_3$ (d) None

128. In how many ways 21 red balls and 19 blue balls can be arranged in a row so that no two blue balls are together?

(a) 1540 (b) 1520 (c) 1560 (d) None

129. In forming a committee of 5 out of 5 males and 6 females how many choices you have to make so that there are 3 males and 2 females?

(a) 150 (b) 200 (c) 1 (d) 461

130. In question No.(129) how many choices you have to make if there are 2 males?

(a) 150 (b) 200 (c) 1 (d) 461

131. In question No.(129) how many choices you have to make if there is no female?

(a) 150 (b) 200 (c) 1 (d) 461

132. In question No.(129) how many choices you have to make if there is at least one female?

(a) 150 (b) 200 (c) 1 (d) 461

133. In question No.(129) how many choices you have to make if there are not more than 3 males?

(a) 200 (b) 1 (c) 461 (d) 401

134. From 7 men and 4 women a committee of 5 is to be formed. In how many ways can this be done to include at least one woman?

(a) 441 (b) 440 (c) 420 (d) None

135. You have to make a choice of 4 balls out of one red one blue and ten white balls. The number of ways this can be done to always include the red ball is ___________.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

136. In question No.(135) the number of ways in which this can be done to include the red ball but exclude the blue ball always is _______.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

137. In question No.(135) the number of ways in which this can be done to exclude both the red and blues ball is _______.

(a) ${}^{11}C_3$ (b) ${}^{10}C_3$ (c) ${}^{10}C_4$ (d) None

138. Out of 6 members belonging to party 'A' and 4 to party 'B' in how many ways a committee of 5 can be selected so that members of party 'A' are in a majority?

(a) 180 (b) 186 (c) 185 (d) 184

139. A question paper divided into 2 groups consisting of 3 and 4 questions respectively carries the note "it is not required to answer all the questions. One question must be answered from each group". In how many ways you can select the questions?

(a) 10 (b) 11 (c) 12 (d) 13

140. The number of words which can be formed with 2 different consonants and 1 vowel out of 7 different consonants and 3 different vowels the vowel to lie between 2 consonants is ______.

(a) $3 \times 7 \times 6$ (b) $2 \times 3 \times 7 \times 6$ (c) $2 \times 3 \times 7$ (d) None

141. How many combinations can be formed of 8 counters marked 1 2 …8 taking 4 at a time there being at least one odd and even numbered counter in each combination?

(a) 68 (b) 66 (c) 64 (d) 62

142. Find the number of ways in which a selection of 4 letters can be made from the word `MATHEMATICS'.

(a) 130 (b) 132 (c) 134 (d) 136

143. Find the number of ways in which an arrangement of 4 letters can be made from the word `MATHEMATICS'.

(a) 1680 (b) 756 (c) 18 (d) 2,454

144. In a cross word puzzle 20 words are to be guessed of which 8 words have each an alternative solution. The number of possible solution is ________.

(a) $(2{\times}8)^2$ (b) ${}^{20}C_{16}$ (c) ${}^{20}C_8$ (d) None

ANSWERS

1.	(c)	**19.**	(b)	**37.**	(a)	**55.**	(b)	**73.**	(b)	**91.**	(a)	**109.**	(c)	**127.**	(c)
2.	(a)	**20.**	(a)	**38.**	(c)	**56.**	(c)	**74.**	(c)	**92.**	(c)	**110.**	(b)	**128.**	(a)
3.	(d)	**21.**	(b)	**39.**	(a)	**57.**	(a)	**75.**	(d)	**93.**	(a)	**111.**	(a)	**129.**	(a)
4.	(a)	**22.**	(c)	**40.**	(c)	**58.**	(a)	**76.**	(d)	**94.**	(b)	**112.**	(c)	**130.**	(b)
5.	(a)	**23.**	(c)	**41.**	(a)	**59.**	(b)	**77.**	(a)	**95.**	(a)	**113.**	(a)	**131.**	(c)
6.	(c)	**24.**	(b)	**42.**	(c)	**60.**	(a)	**78.**	(b)	**96.**	(a)	**114.**	(a)	**132.**	(d)
7.	(a)	**25.**	(a)	**43.**	(a)	**61.**	(b)	**79.**	(d)	**97.**	(b)	**115.**	(c)	**133.**	(d)
8.	(a)	**26.**	(b)	**44.**	(b)	**62.**	(b)	**80.**	(a)	**98.**	(c)	**116.**	(d)	**134.**	(a)
9.	(c)	**27.**	(d)	**45.**	(c)	**63.**	(c)	**81.**	(a)	**99.**	(b)	**117.**	(a)	**135.**	(a)
10.	(a)	**28.**	(a)	**46.**	(c)	**64.**	(d)	**82.**	(a)	**100.**	(c)	**118.**	(d)	**136.**	(b)
11.	(b)	**29.**	(c)	**47.**	(b)	**65.**	(a)	**83.**	(b)	**101.**	(b)	**119.**	(c)	**137.**	(c)
12.	(a)	**30.**	(a)	**48.**	(b)	**66.**	(b)	**84.**	(c)	**102.**	(a)	**120.**	(d)	**138.**	(b)
13.	(a)	**31.**	(a)	**49.**	(a)	**67.**	(c)	**85.**	(b)	**103.**	(b)	**121.**	(c)	**139.**	(c)
14.	(a)	**32.**	(b)	**50.**	(c)	**68.**	(d)	**86.**	(d)	**104.**	(a)	**122.**	(a)	**140.**	(a)
15.	(a)	**33.**	(b)	**51.**	(a)	**69.**	(d)	**87.**	(c)	**105.**	(b)	**123.**	(c)	**141.**	(a)
16.	(a)	**34.**	(a)	**52.**	(b)	**70.**	(a)	**88.**	(a)	**106.**	(a)	**124.**	(c)	**142.**	(d)
17.	(b)	**35.**	(b)	**53.**	(c)	**71.**	(a)	**89.**	(a)	**107.**	(b)	**125.**	(a)	**143.**	(d)
18.	(a)	**36.**	(a)	**54.**	(a)	**72.**	(a)	**90.**	(b)	**108.**	(a)	**126.**	(b)	**144.**	(a)