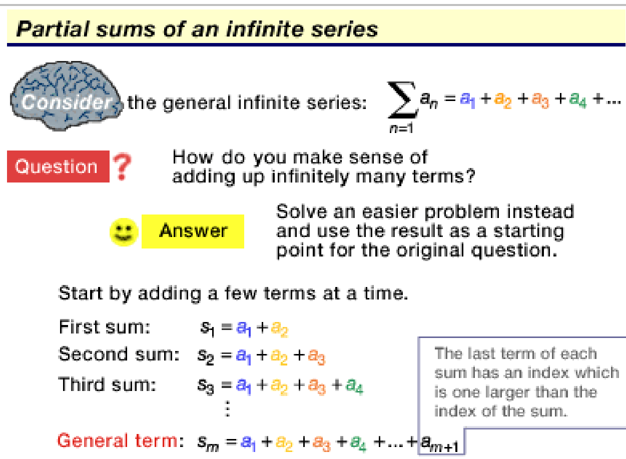
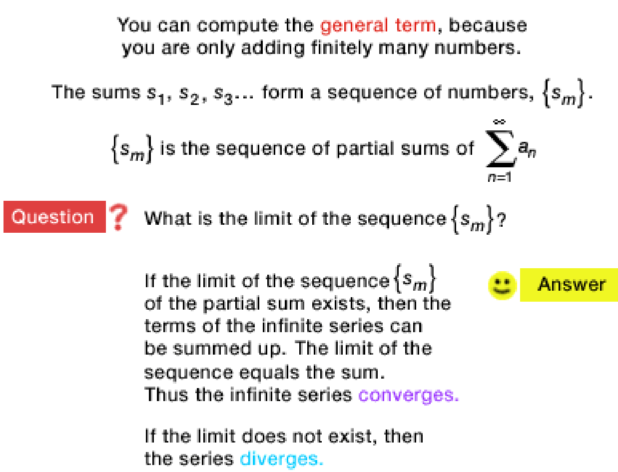
The Summation of Infinite Series

- Given an **infinite series** $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ and the sequence $\{s_m\}$ of **partial sums**, then $\sum_{n=1}^{\infty} a_n$ **converges** if $\lim_{m\to\infty} s_m = S$. In this case, $\sum_{n=1}^{\infty} a_n = S$. The series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\lim_{m\to\infty} s_m$ does not exist.
- You will not be able to determine the sum of most series. However, you can determine whether the series converges or diverges.



To add up an **infinite series**, start by looking at the sequence of **partial sums**. A partial sum is the value you get when you examine only a finite number of terms of the series.

Notice that to evaluate the first partial sum requires two terms of the sequence of the series. In general, to evaluate the *n*th partial sum will require $n + 1$ terms.



Putting each partial sum in an ordered list creates the sequence of partial sums. Notice that the sequence of partial sums tells what the series would equal if it was a finite series.

It is possible to determine what value the sequence of partial sums is approaching by taking a limit.

If the limit of a sequence of partial sums approaches some value, then the sequence has a limit. If the sequence has a limit, the series **converges** to the same limit. This is how you add infinite series.

If the sequence of partial sums does not have a limit, the series **diverges**.

Summation of a well-known series

Example! Look at the following series again:

$$\sum_{n=1}^{\infty}(-1)^{n+1}=1-1+1-1+1...$$

Now look at the associated sequence of partial sums.

odd $s_1=1-1=0$

even $s_2=1-1+1=1$

odd $s_3=1-1+1-1=0$

even $s_4=1-1+1-1+1=1$

Question ? Is this sequence approaching a particular value?

Answer No, the $\lim_{m\to\infty} s_m$ does not exist. Therefore the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}=1-1+1-1+1...$ ← Diverges

Consider this series, sometimes called the devil's series because it perplexed mathematicians for so long.

You saw before how you could argue that the sum of the series was one or zero depending on how you grouped the terms.

Now instead of adding terms, look at the sequence of partial sums.

The sequence of partial sums changes at each term, switching between one and zero.

The sequence does not approach a single value, therefore it does not have a limit. That means the series diverges.

A practical consideration

To determine if an infinite series converges, take the limit of the sequence of partial sums.

$s_1=a_1+a_2$

$s_2=a_1+a_2+a_3$

$s_3=a_1+a_2+a_3+a_4$

$\vdots$

$s_m=a_1+a_2+a_3+a_4+\ldots+a_{m+1}$

Question ? How can you find the sequence of partial sums? And how do you find its limit?

Answer In most cases, it is too complicated to do either. Therefore, you cannot determine what the infinite series sums to.

You can only determine whether the series sums to a number or not.

Although it seems like you could use this approach to find the value of any series, in truth it is kind of hard to add an infinite number of terms together to find out if a series is approaching some value.

A lot of different series problems out there cannot be evaluated directly. But it isn't always necessary to find the summation of the series. Sometimes just knowing that the series converges is enough information. In the following tutorials you will learn about some common series, as well as some ways to test series for convergence.